Università degli Studi Roma Tre
Corso di Laurea in Matematica, a.a. 2014/2015
AL440 - Group Theory
Exercises (April 24^{th}, 2015)

Exercise 1. Let p be a prime and G be a p-group. Let $\{1\} \neq N \triangleleft G$. Show that $N \cap Z(G) \neq \{1\}$.

Exercise 2. Let p be a prime and G be a p-group. Let $\{1\} \neq N \triangleleft G$ such that $\mid N \mid = p$. Show that $N \subseteq Z(G)$.

Exercise 3. Let G be a finite group such that $\mid G \mid = 2k$ with $2 \nmid k$. Show that G contains a subgroup of index 2.

Exercise 4. A group of order p^2q^2, $p < q$, has a normal Sylow subgroup.

Exercise 5. Show that D_6 contains two subgroups ismorphic to S_3.

Exercise 6. A simple group having a subgroup of index n embeds into A_n.

Exercise 7. If G has order 288 or 400 then G is not simple.

Exercise 8. Describe $Aut(\mathbb{Z}_p \times \mathbb{Z}_p)$, for some prime integer p.

Exercise 9. Prove that the group $G = \langle a, b \mid a^2 = 1; b^2 = 1 \rangle$, if it is not abelian then it is infinite. Which kind of group is it?

Exercise 10. Prove that $G = \langle a, b \mid a^4 = 1; a^2 = b^2; bab^{-1} = a^{-1} \rangle$ is the group of quaternions.

Exercise 11. Let p be a prime integer and G be a finite group. Prove that if $G/\Phi(G)$ is a p-group, then G is a p-group.

Exercise 12. In the cyclic group of order 24, $C_{24} = \langle g \rangle$, let $H = \langle g^6 \rangle$ and $K = \langle g^4 \rangle$. Consider the action of K on the set X of all subsets of C_{24} given by the product:

$$\star : K \times X \to X, \quad k \star A = kA.$$

(a) Describe the orbits and the stabilizer of H with respect to $\star$.

(b) Letting K to act on the subsets of G, give a proof of the fact that, if G is a finite group and $H, K \leq G$, then

$$\mid HK \mid = \frac{\mid H \mid\mid K \mid}{\mid H \cap K \mid}$$

Exercise 13. Let G be an abelian finite group with at most n elements of order n, for each $n \geq 1$. Prove that G is cyclic.

Apply this result to show that each finite subgroup of the multiplicative group of a field K is cyclic.