Università degli Studi Roma Tre
Corso di Laurea in Matematica, a.a. 2014/2015
AL440 - Group Theory
Exercises (May 27^{th}, 2015)

Exercise 1. Prove that a group of order 56 is not simple.

Exercise 2. (a) Determine the number of subgroups of the following groups:

- $\mathbb{Z}_{p^n}$
- $\mathbb{Z}_p \times \mathbb{Z}_p$
- $\mathbb{Z}_p \times \mathbb{Z}_{p^2}$

(b) Determine the abelian p-groups with 8 subgroups.

Exercise 3. Show that all groups of order less than 60 are solvable.

Exercise 4. Let be fixed a prime integer p and an integer $n > 1$. Let $A = \langle g \rangle$ be the cyclic group of order p^m and α be the automorphism of A defined by $g \mapsto g^{p+1}$.

(a) Prove by induction on m that $(1+p)^{p^{m-1}} \equiv 1 \pmod{p^m}$.

(b) Prove that α has order p^{m-1} in $Aut(A)$.

(c) Prove that $\langle \alpha \rangle \rtimes A$ is a nilpotent group of class m.

Exercise 5. (a) Prove that a group of order 2107 is solvable.

(b) Determine all groups of order 2107.

Exercise 6. Prove or deny with an example the following statements:

(a) If H and K are normal subgroups of a group G, $H \cong K$, then $G/H \cong G/K$.

(b) If $\alpha \in Aut(G)$ and $N \triangleleft G$, then $G/N \cong G/\alpha(N)$

Exercise 7. Let G be a finite solvable group in which each p-Sylow coincides with its normalizer. Then show that G is a p-group.

Exercise 8. Let G be a finite solvable group in which every maximal subgroup has order coprime with the index. Show that there exists a normal p-Sylow subgroup for some p.