Università degli Studi Roma Tre
Corso di Laurea in Matematica, a.a. 2014/2015
AL440 - Group Theory
Exercises (March 6^{th}, 2015)

Exercise 1. Let N be a normal subgroup of a group G of index n. Show that $g^n \in N$, for each $g \in G$. Using an example, prove that this does not hold if N is not normal in G.

Solution: Each coset gN has order a divisor of n in G/N, thus $(gN)^n = N$ and $g^n \in N$.

Take $G = S_3$ and $N = \{id, (12)\}$ of index 3. Then $(13)^3 = (13) \notin N$.

Exercise 2. Show that in anabelian group the set of finite order elements is a subgroup. What happens in the non abelian case?

Solution: Multipication of finite order elements in an abelian group gives finite order elements (if $xy = yx$ then $o(xy) = mcm(o(x), o(y))$).

If the group is not abelian this may not happen: Take $\mathbb{Z}$ with the following operation:

$$a \circ b = \begin{cases} a + b & \text{se } a \text{ è pari} \\ a - b & \text{se } a \text{ è dispari} \end{cases}$$

Then in $(\mathbb{Z}, \circ)$ there are order 2 elements whose product has infinite order.

Exercise 3. Describe the conjugate classes of in the quaternion group.

Exercise 4. Show that in the dihedral group D_p, where p is prime, all the order 2 elements are conjugated.

Exercise 5. Establishes which claim among the following is true giving a proof or a counterexample:

(a) If the quotien of G on a subgroup of $Z(G)$ is cyclic, then G is cyclic.

(b) If the quotien of G on a subgroup of $Z(G)$ is cyclic, then G is abelian.

(c) If the quotien of G on a subgroup of $Z(G)$ is abelian, then G is abelian.

Exercise 6. Show that in a group G $G \neq HH^g$, for each $g \in G$ (where $H^g = g^{-1}Hg$).

Solution: If $G = HH^g$ for some $g \in G$, then $\exists\ h, h' \in H$ such that $g = hg^{-1}h'g \Rightarrow 1 = hg^{-1}h' \Rightarrow g \in H$. Then $G = HH \Rightarrow G = H$.

Exercise 7. Let G be a group of order 36 and H be a subgroup of G of order 6. Prove that $H \cap H^g \neq \{1\}$ for each $g \in G$.

Solution: Otherwise it would be $G = HH^g$.

Exercise 8. Show that if in a group all the elements have order 2, then the group is abelian.

Solution: It is enough to observe that $xy = (y^{-1}x^{-1})^{-1}$ and that each element coincide with its inverse.

Exercise 9. Let n be a positive integer. Determine the center of the following groups: $\mathbb{Z}_n, S_n, A_n, Q_8$.

Solution: Since $\mathbb{Z}_n$ is abelian, $Z(\mathbb{Z}_n) = \mathbb{Z}_n$.

$Z(S_n) = \{id\}$, if $n \geq 3$. In fact if $id \neq \sigma \in S_n$, then there exist $i \neq j$ such that $\sigma(i) = j$. Then, take $\tau \in S_n$ such that $\tau(i) = i$ and $\tau(j) = k$, for some $k \neq i, j$. It is easy to check $\sigma\tau \neq \tau\sigma$ (in fact $\sigma(\tau(i)) = j$ and $\tau(\sigma(i)) = k$).

As regard A_n, if $n = 3$ $A_3 \cong \mathbb{Z}_3$ and so $Z(A_3) = A_3$. If $n \geq 4$ we can argue as for S_n. If if $id \neq \sigma \in A_n$, then there exist $i \neq j$ such that $\sigma(i) = j$. Take $\tau := (j, k, t) \in A_n$. Again it is easy to check $\sigma\tau \neq \tau\sigma$.

By simply computation it is easy to check that $Z(Q_8)\{1, -1\}$.

Exercise 10. Let H be the subgroup of the dihedral group D_8 generated by an order 2 element a that is not in the center. Determine all the subgroups of D_8 that are conjugate to H, verifying that they are exactly $[D_8 : N(H)]$, where $N(H)$ is the normalizer of H in D_8.