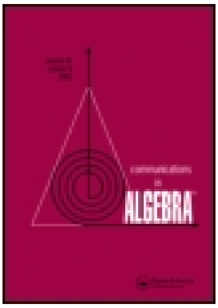

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WHEN THE SEMISTAR OPERATION $\tilde{\star}$ IS THE IDENTITY

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We study properties of integral domains in which it is given a semistar operation $\star$ such that $\tilde{\star}$ is the identity. In particular, we put attention to the case $\star = v$, where v is the divisorial closure.

Key Words: t-Linked overrings; w-Operation; Semistar operation.

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1. INTRODUCTION

Throughout D is an integral domain with quotient field K. To avoid trivial cases we assume that D is not a field. We denote by $\overline{\mathbf{F}}(D)$ the set of nonzero D-modules contained in K, by $\mathbf{f}(D)$ the set of nonzero finitely generated D-modules contained in K and by $\mathbf{F}(D)$ the set of nonzero fractional ideals of D.

A *semistar operation* on D is a map $\star : \overline{\boldsymbol{F}}(D) \to \overline{\boldsymbol{F}}(D)$, $E \mapsto E^\star$, such that, for all $x \in K$, $x \neq 0$, and for all $E, F \in \overline{\boldsymbol{F}}(D)$, the following properties hold:

($\star_1$) $(xE)^\star = xE^\star$;
($\star_2$) $E \subseteq F$ implies $E^\star \subseteq F^\star$;
($\star_3$) $E \subseteq E^\star$ and $E^{\star\star} := (E^\star)^\star = E^\star$ (cf. Fontana and Huckaba, 2000; Okabe and Matsuda, 1994).

In the following we denote by d_D (or simply by d) the identity semistar operation on $\overline{\mathbf{F}}(D)$. We say that a semistar operation is *trivial* if $E^\star = K$ for each $E \in \overline{\mathbf{F}}(D)$ and this happens if and only if $D^\star = K$ (in fact, for each $E \in \overline{\mathbf{F}}(D)$, $E = ED$, whence $E^\star = (ED)^\star \supseteq ED^\star = K$ and so $E^\star = K$).

If $\star$ is a semistar operation on D, we can consider a map $\star_f : \overline{\boldsymbol{F}}(D) \to \overline{\boldsymbol{F}}(D)$ defined as follows: for each $E \in \overline{\boldsymbol{F}}(D)$,

$$E^{\star_f} := \bigcup \left\{ F^\star \mid F \subseteq E, F \in \mathbf{f}(D) \right\}.$$

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It is easy to see that $\star_f$ is a semistar operation on D, called *the semistar operation of finite type associated to* $\star$. Note that, for each $F \in \boldsymbol{f}(D)$, $F^\star = F^{\star_f}$. A semistar operation $\star$ is called a *semistar operation of finite type* if $\star = \star_f$. Moreover $(\star_f)_f = \star_f$ (that is, $\star_f$ is of finite type).

A *quasi–$\star$–ideal* of D is a nonzero ideal I such that $I = I^\star \cap D$. A *quasi–$\star$–prime* is a quasi–$\star$–ideal that is also a prime ideal. A *quasi–$\star$–maximal* ideal is an ideal that is a maximal element in the set of quasi–$\star$–ideals. If $\star$ is a semistar operation of finite type, each quasi-$\star$-ideal is contained in a quasi-$\star$-maximal ideal (Fontana and Huckaba, 2000, Lemma 4.20). We denote by $\star$-Max(D) the set of the quasi-$\star$-maximal ideals of D (when they exist).

Semistar operations are a generalization of the classical concept of star operation described in Gilmer (1972, Section 32). When $D^\star = D$, a semistar operation is usually called a *(semi)star operation* since it is exactly a star operation, when it is restricted to the set of fractional ideals of D. For sake of simplicity, in the following we will refer to a (semi)star operation by simply writing star operation. This is justified by the fact that we will always use semistar operations of finite type (in the sense explained above) and there is a bijection between (semi)star operations of finite type and star operations of finite type on D (Picozza, 2005, Proposition 3.11).

A very simple example of semistar operation that is not a star operation and is not trivial is given by the extension to an overring. Let T be a proper overring of D, $T \neq K$. For each $E \in \overline{\mathbf{F}}(D)$, we put $E^{\star_{\{T\}}} := ET$. Then, $\star_{\{T\}}$ is a semistar operation but not a star operation. In particular, $\star_{\{T\}}$ is of finite type. In fact, for each $E \in \overline{\mathbf{F}}(D)$, we have that

$$E^{\star_{\{T\}}} = ET = \bigcup \left\{ FT \mid F \subseteq E, F \in \mathbf{f}(D) \right\} = \bigcup \left\{ F^{\star_{\{T\}}} \mid F \subseteq E, F \in \mathbf{f}(D) \right\}.$$

A semistar operation $\star$ is *stable* if $(E \cap F)^\star = E^\star \cap F^\star$, for each $E, F \in \overline{\mathbf{F}}(D)$ (cf. Fontana and Huckaba, 2000). For example, the semistar operation $\star_{\{T\}}$ is stable if and only if T is a flat overring of D (Picozza, 2005, Proposition 1.2).

Given a semistar operation $\star$ on an integral domain D it is possible to construct a semistar operation $\tilde{\star}$ which is stable and of finite type defined as follows: for each $E \in \overline{\mathbf{F}}(D)$,

$$\begin{aligned} E^{\tilde{\star}} &:= \left\{ x \in K \mid xJ \subseteq E, \text{ for some } J \subseteq D, J \in \mathbf{f}(D), J^\star = D^\star \right\} \\ &= \bigcup \left\{ (E : J) \mid J \subseteq D, J \in \mathbf{f}(D), J^\star = D^\star \right\}. \end{aligned}$$

An equivalent definition of the operation $\tilde{\star}$ is the following:

$$E^{\tilde{\star}} := \bigcap_{P \in \star_f - \mathrm{Max}} ED_P.$$

It is well known that $\tilde{\star}$ is a semistar operation and $\star_f$-Max$(D) = \tilde{\star}$-Max(D) (Fontana and Loper, 2003, Corollary 3.5(2)). For a more detailed account about $\tilde{\star}$ we refer the reader to Wang and McCasland (1997), Anderson and Cook (2000), Fontana and Huckaba (2000), and Fontana and Loper (2003). We recall the following well-known fact.

Lemma 1.1 (Fontana and Huckaba, 2000, Corollary 3.9(2)). *Let D be a domain with a semistar operation $\star$. Then $\star = \tilde{\star}$ if and only if $\star$ is stable and of finite type.*

If $\star_1$ and $\star_2$ are two semistar operations on D, we say that $\star_1 \leq \star_2$ if $E^{\star_1} \subseteq E^{\star_2}$, for each $E \in \overline{\mathbf{F}}(D)$. In general we have that $\tilde{\star} \leq \star_f \leq \star$, for any semistar operation $\star$.

A particularly interesting case is when $\star = v$.

We briefly recall that if I is a nonzero fractional ideal of D, then $I^v := (I^{-1})^{-1} = (D : (D : I))$ is the *divisorial closure* of I, which is a star operation, and

$$I^{v_f} = \bigcup \left\{ J^v \mid J \subseteq I, J \in \mathbf{f}(D) \right\} =: I^t,$$

is the *t-closure* of I. We say that I is *divisorial* (or a *v-ideal*) if $I = I^v$ and that I is a *t-ideal* if $I = I^t$. In general, $I \subseteq I^t \subseteq I^v$ (in particular, a divisorial ideal is a t-ideal). The t-operation is the finite type star operation associated with v.

A nonzero ideal J of D is called a *Glaz–Vasconcelos ideal* (in short, a GV-ideal) if J is finitely generated and $J^{-1} = D$. The set of GV-ideals of D is denoted by $\mathrm{GV}(D)$.

Given a nonzero fractional ideal I of D, the *w-closure* of I is the ideal

$$I^w = \{x \in K \mid xJ \subseteq I \text{ for some } J \in \mathrm{GV}(D)\},$$

that is, $w = \tilde{v}$.

A nonzero fractional ideal I is a w-ideal if $I = I^w$. Now, $w \leq t \leq v$, that is, if $I \in \mathbf{F}(D)$, we have the following inclusions:

$$I \subseteq I^w \subseteq I^t \subseteq I^v.$$

It is well known that if $\star$ is a star operation on D, then $\star \leq v$, $\star_f \leq t$ and $\tilde{\star} \leq w$.

The structure of the set of $\star$-ideals in a domain D, for a chosen semistar operation $\star$, generally reflects some important properties of D. Thus, investigations in this area may play an interesting role in ring classification problems.

For instance, in the more or less recent literature (cf. Bass, 1962; Bazzoni, 2000; Bazzoni and Salce, 1996; Heinzer, 1968), domains in which every nonzero ideal is divisorial (i.e., *divisorial domains*) have been widely studied and characterized in the integrally closed case. Also, it is well known that Prüfer domains are exactly the integrally closed domains in which $t = d$ (Gilmer, 1972, Proposition 34.12).

There are interesting results about domains in which every t-ideal is divisorial (TV-domains, Houston and Zafrullah, 1988) and domains in which every w-ideal is divisorial (El Baghdadi and Gabelli, 2005).

Moreover, very recently, Mimouni (2005) has studied domains in which each nonzero fractional ideal of a domain D is a w-ideal, that is, where the operation w is the identity (*DW-domains*).

Since DW-domains are exactly the domains in which proper GV-ideals do not exist (Corollary 2.6), we have that the DW-property is equivalent to say that each finitely generated, integral ideal is contained in a proper, integral, divisorial ideal.

This fact well generalizes the behavior with respect to the v-operation of the:

- Domains in which $t = d$: these are exactly the ones in which each finitely generated ideal is divisorial;
- Domains in which $v = d$: these are exactly the ones in which each ideal is divisorial.

Now, we have observed that DW-domains were already known some years before (Dobbs et al., 1989, 1990, 1992) under the name of *t-linkative domains*. We briefly recall that an overring T of a domain D is t-linked over D if for each nonzero finitely generated fractional ideal J of D, such that $J^t = D$, then $(JT)^t = T$. A domain D is t-linkative if every overring of D is t-linked over D. Now, Dobbs et al. (1990, Theorem 2.6) establishes that a domain D is t-linkative if and only if every maximal ideal is a t-ideal. This is exactly the characterization of DW-domains given in Mimouni (2005, Proposition 2.2).

In Section 2 we consider a generic semistar operation $\star$ on a domain D and study when $\tilde{\star} = d$, so generalizing the concept of DW-domain and some of the results proved in Mimouni (2005). In particular we will see that the condition $\tilde{\star} = d$ is related to properties of valuation overrings of D, especially to a generalization of the t-linkedness condition (Theorem 2.15).

In Section 3, we deal with the particular case of DW-domains and we study when the DW-property on D transfers to the integral closure D'. We are interested in evaluating how far integrally closed, DW-domains are from being Prüfer. This question naturally arises from the fact that in literature there are various results concerning the study of the Prüfer property for the integral closure of a domain D. We show that all these known results are based on the fact that the domain D is just DW. We will also study when the DW-property is local and give some results in order to characterize the quasilocal (and semiquasilocal) DW-domains.

Finally, in Section 4 we will investigate the DW-property in Mori and, in particular, Noetherian domains.

2. WHEN $\tilde{\star}$ IS THE IDENTITY

Generalizing a well known terminology on star operations (Gilmer, 1972, p. 395), we say that two semistar operations $\star_1, \star_2$ on a domain D are *equivalent* if $(\star_1)_f = (\star_2)_f$. In this case we write $\star_1 \sim_f \star_2$. Some important classes of integral domains are characterized by the properties of systems of ideals determined by semistar operations. For some of these characterizations it is enough to control the behavior of the semistar operations on the finitely generated ideals, that is, they only depend on the equivalence classes of semistar operations with respect to $\sim_f$. For example, given a semistar operation $\star$ on a domain D, D is $\star$-Noetherian (i.e., the ascending chain condition, a.c.c., property on the quasi-$\star$-ideals of D holds) if and only if D is $\star_f$-Noetherian (El Baghdadi et al., 2004, Proposition 3.5). Moreover, the behavior of a domain D with respect to a semistar operation $\star$ may only depend on $\tilde{\star}$. This happens, for instance, for the property of being P$\star$MD or semistar Dedekind (cf. El Baghdadi et al., 2004; Fontana et al., 2003). This leads to the following definition.

Definition 2.1. Two semistar operations $\star_1, \star_2$ on a domain D are *weakly equivalent* if $\widetilde{\star_1} = \widetilde{\star_2}$. In this case, we write $\star_1 \sim_w \star_2$.

It is easily seen from the definitions that when $\star_1 \sim_f \star_2$ then $\star_1 \sim_w \star_2$. The converse is not true. In fact, $v \sim_w w$, but, in general, $t \not\sim_f w$ ($v_f = t$ and $w_f = w$).

We also notice that $\star_f \sim_w \tilde{\star}$ (Fontana and Huckaba, 2000, Proposition 3.6(b)), for any semistar operation $\star$.

Proposition 2.2. *Given a domain D, the following conditions are equivalent*:

(i) *D is a Prüfer domain*;
(ii) *For any two semistar operations $\star_1, \star_2$ on D*

$$\star_1 \sim_w \star_2 \Rightarrow \star_1 \sim_f \star_2,$$

that is, $\sim_f = \sim_w$;
(iii) *For any semistar operation $\star$ on D*, $\tilde{\star} = \star_f$.

Proof. (i) $\Rightarrow$ (ii) By Picozza (2005, Lemma 4.4) in a Prüfer domain D each semistar operation of finite type is the extension to an overring of D. From Fontana et al. (1997, Theorem 1.1.1), every overring of a Prüfer domain is flat. So any semistar operation of finite type is stable by Picozza (2005, Proposition 1.2) and $(\star_1)_f = \widetilde{\star_1} = \widetilde{\star_2} = (\star_2)_f$.

(ii) $\Rightarrow$ (iii) For any semistar operation $\star$, $\star_f \sim_w \tilde{\star}$. From (ii), we have that $\star_f \sim_f \tilde{\star}$. Then $\star_f = \tilde{\star}$, since they both are semistar operations of finite type.

(iii) $\Rightarrow$ (i) Statement (iii) means that any semistar operation on D of finite type is stable by Lemma 1.1. In particular any extension to an overring of D is stable, because it is of finite type. Hence each overring of D is flat and D is a Prüfer domain (Fontana et al., 1997, Theorem 1.1.1; Picozza, 2005, Proposition 1.2).

We recall that given an integral domain D with a semistar operation $\star$, D is a P$\star$MD if its localizations at the quasi-$\star_f$-prime ideals are valuation domains and that D is $\star$-Dedekind if it is $\star$-Noetherian and its localizations at the quasi-$\star_f$-prime ideals are rank-one discrete valuation domains (DVR).

By Fontana et al. (2003, Theorem 3.1) and El Baghdadi et al. (2004, Corollary 4.3) we have the following result.

Proposition 2.3. *If $\star_1, \star_2$ are two weakly equivalent semistar operations on a domain D, then D is a P$\star_1$MD if and only if D is a P$\star_2$MD and D is a $\star_1$-Dedekind domain if and only if D is a $\star_2$-Dedekind domain.*

However, there are cases in which the weakly equivalence between two semistar operations does not induce the equivalence between properties of the domain related to the operations themselves. For instance, $t \sim_w w$, but the classes of Mori domains (t-Noetherian) and Strong Mori domains (w-Noetherian) are distinct.

Proposition 2.4. *Let D be a domain with two semistar operations, $\star_1$ and $\star_2$. The following conditions are equivalent*:

(i) $\star_1 \sim_w \star_2$;
(ii) *The set of quasi-$(\star_1)_f$-maximal ideals of D coincides with the set of quasi-$(\star_2)_f$-maximal ideals*;
(iii) *For each nonzero ideal I of D, $I^{(\star_1)_f} = D^{\star_1}$ if and only if $I^{(\star_2)_f} = D^{\star_2}$.*

Proof. (i) $\Rightarrow$ (ii) It follows immediately from the fact that for any semistar operation $\star$, the set of quasi-$\star_f$-maximal ideals coincides with the set of quasi-$\tilde{\star}$-maximal ideals (Fontana and Loper, 2003, Corollary 3.5(2)).

(ii) $\Rightarrow$ (iii) By Fontana and Huckaba (2000, Lemma 4.20), for any semistar operation $\star$, $I^{\star_f} \subsetneq D^\star$ if and only if I is contained in a quasi-$\star_f$-maximal ideal.

(iii) $\Rightarrow$ (i) It is a consequence of the definition of $\tilde{\star}$. □

In the following we will be interested in studying when a semistar operation $\star$ on D is weakly equivalent to the identity d, that is, when $\tilde{\star} = d$.

Corollary 2.5. *Let D be a domain and let $\star$ be a semistar operation on D. The following conditions are equivalent*:

(i) $\tilde{\star} = d$;
(ii) *Each maximal ideal of D is quasi-$\star_f$-maximal*;
(iii) *If $I \neq D$ is a nonzero finitely generated ideal of D, then $I^{\star_f} \subsetneq D^{\star_f}$.*

The version of Corollary 2.5 for $\star = v$ gives the characterization of DW-domains in Mimouni (2005, Proposition 2.2). We denote by $\mathrm{Max}(D)$ the set of the maximal ideals of D and by t-$\mathrm{Max}(D)$ the set of the t-maximal ideals of D.

Corollary 2.6. *Let D be a domain. The following conditions are equivalent*:

(i) *D is a DW-domain*;
(ii) *Every maximal ideal of D is a t-ideal* (*i.e.*, $\mathrm{Max}(D) = t\text{-}\mathrm{Max}(D)$);
(iii) $\mathrm{GV}(D) = \{D\}$.

As we pointed out in the Introduction, the problem of characterizing domains in which every ideal is divisorial has been investigated in different contexts. Since the divisorial closure is the unique maximal star operation, in any domain D, it follows that divisorial domains are exactly the domains in which any star operation is the identity. Analogously, as being the t-operation the unique, maximal star operation of finite type, domains in which the t-operation is the identity are exactly the domains in which any star operation of finite type is the identity. Also, for the maximality of w among the stable and finite type star operations in any domain D, DW-domains are the domains in which $\tilde{\star} = d$ for any star operation $\star$.

Theorem 2.7 (cf. Okabe and Matsuda, 1994, Theorem 48). *Let D be a domain. The following conditions are equivalent*:

(i) *D is a DVR*;
(ii) *$\star = d$, for any nontrivial semistar operation $\star$ on D*;
(iii) *$\star_f = d$, for any nontrivial semistar operation $\star$ on D.*

Proof. (i) $\Rightarrow$ (ii), (iii). If D is a DVR, then every nontrivial semistar operation on D is a star operation, because if $D^\star$ is a proper overring of D, then $D^\star = K$ and so $\star$ is trivial. Hence $\star \leq v$. Since $v = d$ (Gilmer, 1972, §34, Ex. 12), it follows that $\star = d$ and also $\star_f = d$. Conversely, suppose that (ii) holds. If $T \neq K$ is an overring of D, then $\star_{\{T\}}$ is a nontrivial semistar operation on D different from d. Thus, D cannot have overrings distinct from K, and so D is a one-dimensional valuation domain. Again, from Gilmer (1972, §34, Ex. 12) V is a DVR. Since $\star_{\{T\}}$ is a semistar operation of finite type it is also proved (iii) $\Rightarrow$ (i). □

Theorem 2.8. *Let D be a domain. The following conditions are equivalent*:

(i) *D is one-dimensional and quasilocal*;
(ii) *$\tilde{\star} = d$, for any nontrivial semistar operation $\star$ on D.*

Proof. First we note that for any nontrivial semistar operation $\star$ there exist proper quasi-$\star$-ideals. In fact, if $D^\star \neq K$, there exists $x \in D\backslash\{0\}$ such that x is not invertible in $D^\star$. Then, it is easy to see that $xD^\star \cap D$ is a proper quasi-$\star$-ideal. So, by Fontana and Huckaba (2000, Lemma 4.20), each nontrivial semistar operation has quasi-$\star_f$-maximal ideals. Now, assume that D is one-dimensional and quasilocal with maximal ideal M and let $\star$ be a nontrivial semistar operation on D. Since D has quasi-$\star_f$-maximal ideals and a quasi-$\star_f$-maximal ideal is prime, it follows that M is a quasi-$\star_f$-maximal ideal (since it is the only nonzero prime ideal). It follows that $\tilde{\star} = d$ (Corollary 2.5).

Conversely, from Picozza (2005, Proposition 1.2), if T is a flat overring of D, then $\star_{\{T\}} = \widetilde{\star_{\{T\}}}$. It follows that D cannot have proper flat overrings and this is equivalent to require that D is one-dimensional and quasilocal, since each localization to a prime ideal is a flat overring of D. □

In Mimouni (2005, Corollary 2.3) it is shown that if D is a domain of t-dimension 1, then D is DW if and only if D is one-dimensional. We can also easily show the following result.

Proposition 2.9. *Let D be a one-dimensional domain. Then D is* DW.

Proof. By Jaffard (1960, Corollaire 3, p. 31) a prime ideal which is minimal over a t-ideal is t-prime. Let M be a maximal ideal of D. Then M is minimal over a nonzero principal ideal and so it is a t-ideal. It follows that D is DW. □

Theorem 2.10. *Let D be a DW-domain with a semistar operation $\star$. The following conditions are equivalent*:

(i) *$\tilde{\star} = d$*;
(ii) *$D^{\tilde{\star}} = D$.*

Proof. (i) $\Rightarrow$ (ii) It is obvious.

(ii) $\Rightarrow$ (i) If $D^{\tilde{\star}} = D$, then $\tilde{\star}$ is a star operation, whence $\tilde{\star} \leq w$. But $w = d$ and the thesis follows. □

Combining Proposition 2.9 and Theorem 2.10, we have the following corollary.

Corollary 2.11. *Let D be a one-dimensional domain with a semistar operation $\star$. Then $\tilde{\star} = d$ if and only if $D^{\tilde{\star}} = D$.*

Remark 2.12. The analog of Proposition 2.9 holds for any $\tilde{\star}$, when $\star$ is a star operation (that is, if D is one dimensional, then $\tilde{\star} = d$ for each star operation $\star$), but, as it is shown in Corollary 2.11, this result cannot be extended to proper semistar operations. For instance, consider a one-dimensional domain D with two maximal ideals, M_1 and M_2 and the semistar operation $\star := \star_{\{D_{M_1}\}}$. Since D_{M_1} is a flat overring of D, $\tilde{\star} = \star$ by Picozza (2005, Proposition 1.2), and so $\star \neq d$ since $D^{\tilde{\star}} = D_{M_1} \neq D$.

As mentioned in the Introduction, DW-domains are exactly the t-linkative domains. We can give an analogous characterization for the domains in which $\tilde{\star} = d$, where $\star$ is a semistar operation on D. The linkedness property for an overring of D, with respect to semistar operations, has been studied by El Baghdadi and Fontana (2004). We recall that, if $\star$ is a semistar operation on a domain D and $\star'$ is a semistar operation on an overring T of D, then T is $(\star, \star')$-linked over D if $I^\star = D^\star$ implies $(IT)^{\star'} = T^{\star'}$, for each nonzero finitely generated ideal I of D. In this more general context, the t-linkedness corresponds exactly to the (t_D, t_T)-linkedness, where t_D and t_T represent, respectively, the t-operation on D and on T. We notice that the $(\star, \star')$-linkedness property of the overring T only depends on the semistar operations $\tilde{\star}$ and $\tilde{\star'}$. More precisely, by El Baghdadi and Fontana (2004, Theorem 3.8), if $\star_1$ is a semistar operation on D and $\star_2$ is a semistar operation on T such that $\star_1 \sim_w \star$ and $\star_2 \sim_w \star'$, then T is $(\star, \star')$-linked over D if and only if T is $(\star_1, \star_2)$-linked over D.

Lemma 2.13. *Let D be a domain with a semistar operation D. Let T be an overring of D. The following conditions are equivalent*:

(i) $T^{\tilde{\star}} = T$;
(ii) T *is* $(\star, t_T)$*-linked over* D.

Proof. (i) $\Rightarrow$ (ii) Let J be a nonzero finitely generated ideal of D such that $J^\star = D^\star$. Let $x \in (T : JT)$, then $xT \subseteq (T : J)$ and so $xT \subseteq T^{\tilde{\star}} = T$. It follows that $x \in T$ and so $(JT)^{-1} = (T : JT) \subseteq T$. Hence, $(JT)^{t_T} = (JT)^{v_T} = T$ (where v_T denotes the v-operation on T) and T is $(\star, t_T)$-linked over D.

(ii) $\Rightarrow$ (i) By hypothesis we have that if $I \in \mathbf{f}(D)$ and $I^\star = D^\star$, then $(IT)^{t_T} = T$. It follows that,

$$T^{\tilde{\star}} = \bigcup \{(T : I) \mid I \subseteq D, I \in \mathbf{f}(D), I^\star = D^\star\}$$

$$\subseteq \bigcup \{(T : J) \mid J \subseteq T, J \in \mathbf{f}(T), J^{v_T} = J^{t_T} = T\}$$
$$= T^{w_T} = T,$$

where w_T is the w-operation on T. □

If T is an overring of a domain D and $\star$ is a semistar operation on D, we denote by $\star|_{\overline{\mathbf{F}}(T)}$ the restriction of $\star$ to $\overline{\mathbf{F}}(T)$. This is a semistar operation on T by Okabe and Matsuda (1994, Lemma 45). We note that if $T^\star = T$, then $\star|_{\overline{\mathbf{F}}(T)}$ is a star operation on T. In particular, if T is t-linked over D, then $w|_{\overline{\mathbf{F}}(T)}$ is a star operation. Also, it is easy to check that $w|_{\overline{\mathbf{F}}(T)}$ is stable and of finite type.

Proposition 2.14. *Let D be a domain with a semistar operation $\star$. The following conditions are equivalent*:

(i) *P is a quasi-$\tilde{\star}$-prime ideal of D*;
(ii) *$\tilde{\star}|_{\overline{\mathbf{F}}(V)} = d_V$, for each valuation overring of D centered in P*;
(iii) *$V^{\tilde{\star}} = V$ for each valuation overring of D centered in P*;
(iv) *There exists a valuation overring V of D centered in P such that $V^{\tilde{\star}} = V$.*

Proof. (i) ⇒ (ii) Let V be a valuation overring of D with maximal ideal M such that $M \cap D = P$. Let $* := \tilde{\star}_{|\overline{F}(V)}$. Suppose that $* \neq d_V$. Since $*$ is a semistar operation of finite type on V (Fontana and Loper, 2001, Corollary 2.8) distinct from the identity, it is the extension to a proper overring W of V (see for example Picozza, 2005, Lemma 4.4). So, $M^{\tilde{\star}} \cap V = M^* \cap V = MW \cap V = W \cap V = V$. Thus, $P^{\tilde{\star}} \cap D = (M \cap D)^{\tilde{\star}} \cap D = M^{\tilde{\star}} \cap D = M^{\tilde{\star}} \cap V \cap D = V \cap D = D$, a contradiction, since P is a quasi-$\tilde{\star}$-ideal of D. Hence, $* = d_V$.

(ii) ⇒ (iii) ⇒ (iv) is obvious.

(iv) ⇒ (i) Let M be the maximal ideal of V and let $* := \tilde{\star}_{|\overline{F}(V)}$. Since $*$ is a star operation of finite type on V, $M^{\tilde{\star}} = M^* = M$. It follows that $P^{\tilde{\star}} \cap D = (M \cap D)^{\tilde{\star}} \cap D = M^{\tilde{\star}} \cap D = M \cap D = P$. Hence, P is a quasi-$\tilde{\star}$-ideal of D. □

Theorem 2.15 (cf. El Baghdadi and Fontana, 2004, Theorem 3.9). *Let D be a domain with a semistar operation $\star$. The following conditions are equivalent*:

(i) $\tilde{\star} = d$;
(ii) *$T^{\tilde{\star}} = T$ for each overring T of D*;
(ii)′ *Each overring T of D is $(\star, t_T)$-linked over D*;
(iii) *$V^{\tilde{\star}} = V$ for each valuation overring V of D*;
(iii)′ *Each valuation overring V of D is $(\star, t_V)$-linked over D*;
(iv) *$V^{\tilde{\star}} = V$ for each valuation overring V of D centered in a maximal ideal of D*;
(iv)′ *Each valuation overring V of D centered in a maximal ideal of D is $(\star, t_V)$-linked over D*;
(v) *For each maximal ideal M of D, there exists a valuation overring V of D centered in M such that $V^{\tilde{\star}} = V$*;
(v)′ *For each maximal ideal M of D, there exists a valuation overring V of D centered in M that is $(\star, t_V)$-linked over D.*

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) is obvious. (ii) $\Leftrightarrow$ (ii)$'$, (iii) $\Leftrightarrow$ (iii)$'$, (iv) $\Leftrightarrow$ (iv)$'$ and (v) $\Leftrightarrow$ (v)$'$ follow immediately by Lemma 2.13.

(v) $\Rightarrow$ (i) is a consequence of Proposition 2.14(iv) $\Rightarrow$ (i) and Corollary 2.5(ii) $\Rightarrow$ (i). $\square$

By the fact that $\tilde{t} = w$ we immediately recover Dobbs et al. (1992, Lemma 2.1) as a corollary of the previous theorem (using the equivalence (i) $\Leftrightarrow$ (ii)$'$).

Corollary 2.16. *Let D be an integral domain. Then D is DW if and only if each overring of D is t-linked over D.*

We recall that the intersection of t-linked overrings of D is still t-linked over D and each generalized quotient ring (in particular each localization) of D is t-linked over D (Dobbs et al., 1989, Proposition 2.2(b), (d)). Thus, if D is a *GQR-domain* (i.e., each overring of D is an intersection of generalized quotient rings of D), then D is DW. In particular QQR-domains, studied by Gilmer and Heinzer (1967), are DW.

A well-known semistar operation is the b-operation (the "completion" or "integral closure" of ideals), defined as follows: let D be a domain and let $\{V_\alpha\}_{\alpha\in A}$ be the set of the valuation overrings of D. If $E \in \overline{\mathbf{F}}(D)$, then the b-closure of E is $E^b := \bigcap_{\alpha\in A} EV_\alpha$. It is easy to see that b is a star operation if and only if D is integrally closed (and this is the case considered in Gilmer (1972, §32), otherwise b is a proper semistar operation.

Corollary 2.17. *Let D be a domain. Then $\tilde{b} = d$. In particular, for any semistar operation $\star$ on D, $\tilde{\star} = d$ if and only if $\tilde{\star} \leq b$.*

Proof. If V is an overring of D, then $V^{\tilde{b}} \subseteq V^b = V$, whence $V = V^{\tilde{b}}$. Thus, using the condition given in Theorem 2.15(iii) the thesis follows. $\square$

Now, we ask whether the property $\star = d$, for some semistar operation $\star$ on a domain D, is a local property. Given a prime ideal P of D, we put $\star_P := \star|_{\overline{\mathbf{F}}(D_P)}$.

Proposition 2.18. *Let D be a domain D with a semistar operation $\star$. The following conditions are equivalent*:

(i) $\star = d$;
(ii) $\star_M = d_M$, *for each maximal ideal M of D.*

Proof. (i) $\Rightarrow$ (ii) It is obvious.

(ii) $\Rightarrow$ (i) This directly follows from the fact that $E = \bigcap_{M\in\mathrm{Max}(D)} E_M$ and

$$E^\star \subseteq \bigcap_{M\in\mathrm{Max}(D)} (E_M)^\star = \bigcap_{M\in\mathrm{Max}(D)} (E_M)^{\star_M} = \bigcap_{M\in\mathrm{Max}(D)} E_M = E. \qquad \square$$

3. DW-DOMAINS

In this section we study domains in which the w-operation is the identity (DW-domains) and outline some relations between the DW-property of D and of D' (where D' denotes, as usual, the integral closure of D). In Dobbs et al. (1992, Corollary 2.7) it is shown that treed domains are always DW. Important classes of treed domains are Prüfer domains, domains with Krull dimension ≤ 1, going-down domains (these classes are explicitely considered in Dobbs et al., 1992), and the stable domains (Sally and Vasconcelos, 1974). Moreover, Dobbs et al. (1992, Theorem 2.4) shows that domains whose integral closure is Prüfer are DW-domains, and these are not necessarily treed (for such an example see, for instance, Papick, 1976b, Example 2.28).

In the following, we will give examples of DW-domains which do not belong to any of the above classes of domains.

We recall that given a domain D the Nagata Ring (see, for instance, Gilmer, 1972, §33) is defined as follows:

$$D(X) := \{f/g \mid f, g \in D[X], c(g) = D\},$$

(where $c(h)$ is the content of a polynomial $h \in D[X]$). First in Kang (1989) and then in Fontana and Loper (2003), the authors extended the construction of the Nagata Ring referring to an arbitrary chosen semistar operation $\star$ on D, as follows:

$$Na(D, \star) := \{f/g \mid f, g \in D[X], g \neq 0, c(g)^{\star} = D^{\star}\}.$$

With these notation $Na(D, d) = D(X)$. We also recall that $Na(D, \star) = Na(D, \star_f) = Na(D, \tilde{\star})$ (Fontana and Loper, 2003, Corollary 3.5).

We remark that $Na(D, v)$ is always DW. In fact the maximal ideals of $Na(D, v)$ are of the type $M(X)$, where M ranges among the t-maximal ideals of D (Kang, 1989, Proposition 2.1). Thus they are t-ideals from Kang (1989, Corollary 2.3). In particular, we have the following proposition.

Proposition 3.1. *Let D be a domain. The following conditions are equivalent*:

(i) *D is DW*;
(ii) *$Na(D, v) = D(X)$*;
(iii) *$D(X)$ is DW.*

Proof. (i) $\Rightarrow$ (ii) $Na(D, v) = Na(D, w) = Na(D, d) = D(X)$.

(ii) $\Rightarrow$ (iii) Obvious, since we have observed above that $Na(D, v)$ is always DW.

(iii) $\Rightarrow$ (i) Since the maximal ideals of $D(X)$ are exactly the ideals $M(X)$, with $M \in \mathrm{Max}(D)$, and since $M(X)^t = M^t(X)$ (Kang, 1989, Corollary 2.3), the thesis directly follows. □

So, the Nagata Ring can be used to give new examples of DW-domains. For instance, it is known that $D(X)$ is treed if and only if D is treed and the integral closure of D is Prüfer (Anderson et al., 1989, Theorem 2.10). Thus if we take a treed domain D such that D' is not Prüfer (for instance, take $D := \mathbb{Q} + X\mathbb{Q}(Y)[[X]]$; Anderson et al., 1989, Remark 2.11), then $D(X)$ is DW but not treed. Moreover the integral closure $D(X)' = D'(X)$ (Anderson et al., 1989, Proposition 2.6) is not Prüfer (Arnold, 1969).

Heinzer (1968) showed that the divisorial integrally closed domains are the Prüfer domains satisfying determined finiteness conditions on prime ideals. Also, it is known that the integrally closed domains such that $t = d$ are exactly the Prüfer domains. But, for a general integral domain, these questions are still open.

Now, the class of integrally closed DW-domains obviously contains the Prüfer domains as a proper subset. We will show that this class also contains a family of pullback domains, but it is even larger because we will give examples of integrally closed DW-domains which are neither Prüfer, nor a pullback belonging to the above-cited family.

Proposition 3.2. *Let D be a domain such that every maximal ideal M is not the union of prime ideals properly contained in M. Then D and D' are DW-domains.*

Proof. Since any maximal ideal M is not the union of prime ideals properly contained in M, then M is minimal over a principal ideal. Thus, M is a t-ideal (Jaffard, 1960, Corollaire 3, p. 31) and D is DW. It is easy to check that if each maximal ideal of D is minimal over a principal ideal, then the same holds for each maximal ideal of D'. Thus D' is DW too. □

We notice that there exist DW-domains in which each maximal ideal is a union of prime ideals. For example take a valuation domain with the maximal ideal unbranched (Gilmer, 1972, Theorem 17.3).

We have the following corollary.

Corollary 3.3. *Let D be a domain with finite prime spectrum. Then D and D' are DW-domains.*

In general, if a domain D satisfies the following property:

($\Diamond$) each maximal ideal is minimal over a principal ideal,

then D is DW (since each maximal ideal is a t-ideal). Moreover, if D satisfies ($\Diamond$), then D' satisfies ($\Diamond$) too and so D' is DW. This is exactly the argument proving Proposition 3.2 and Corollary 3.3. If we consider a treed domain of finite dimension, then it is easy to check that ($\Diamond$) holds. Hence finite-dimensional, treed domains form another class of DW-domains whose integral closure is DW.

But, generally, the integral closure of a DW-domain is not necessarily DW. Consider the following pullback diagram:

$$\begin{array}{ccc} D = \mathbb{R} + (X, Y)\mathbb{C}[X, Y]_{(X,Y)} & \longrightarrow & \mathbb{R} \\ \downarrow & & \downarrow \\ \mathbb{C}[X, Y]_{(X,Y)} & \longrightarrow & \mathbb{C}. \end{array}$$

Here and in the following diagrams, the vertical arrows are the natural inclusions and the horizontal arrows are the natural projections. Then, D is DW by Dobbs et al. (1989, Remark 2.8(b)). The integral closure of D is $D' = \mathbb{C}[X, Y]_{(X,Y)}$ (Fontana, 1980, Corollary 1.5) and this is not DW. In fact D' is a quasilocal, Krull domain (being Noetherian and integrally closed). Thus D' is DW if and only if M is principal (Mimouni, 2005, Corollary 2.3), which does not hold.

Proposition 3.4. *Let D be a domain. The following conditions are equivalent*:

(i) *D is DW*;
(ii) $w|_{\overline{\mathbf{F}}(D')} = d_{D'}$;
(iii) *There exists an overring T of D such that the Lying Over* (LO) *property holds for the extension $D \subset T$ and $w|_{\overline{\mathbf{F}}(T)} = d_T$.*

Proof. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are obvious.

(iii) $\Rightarrow$ (i) Let M be a maximal ideal of D. Since the LO holds, there exists a prime ideal P of T such that $P \cap D = M$. Then

$$M^w = (P \cap D)^w = P^w \cap D = P^{d_T} \cap D = P \cap D = M.$$

Hence M is a w-ideal, thus M is a t-ideal (since t-Max$(D) = w$-Max(D)) and D is DW. □

Corollary 3.5. *Let D be a domain. If there exists an overring T of D which is an integral extension of D, t-linked over D and it is DW, then D is DW.* (*In particular, if D' is t-linked over D and DW then D is DW.*)

Proof. Since T is t-linked over D, then $w|_{\overline{\mathbf{F}}(T)}$ is a star-operation on T, stable and of finite type (as we have noticed at p. 1962). Then, $w|_{\overline{\mathbf{F}}(T)} \leq w_T = d_T$ and so $w|_{\overline{\mathbf{F}}(T)} = d_T$. Moreover since $D \subset T$ is an integral extension, the LO holds and D is DW by Proposition 3.4. □

The converse of Corollary 3.5 does not hold. In fact if D is DW and D' is t-linked over D, then D' may not be DW. An example of this fact is given again by the domain $D = \mathbb{R} + (X, Y)\mathbb{C}[X, Y]_{(X,Y)}$. In fact we have seen that $D' = \mathbb{C}[X, Y]_{(X,Y)}$ is not DW and D is DW. Moreover D' is t-linked over D since we have already noticed that all the overrings of a DW-domain are t-linked.

It is well known that if $D \subset R \subset T$ are integral domains (with the same quotient field) such that R is t-linked over D and T is t-linked over R, then T is t-linked over D, that is,

$$D \overset{t\text{-linked}}{\hookrightarrow} R \overset{t\text{-linked}}{\hookrightarrow} T \Rightarrow D \overset{t\text{-linked}}{\hookrightarrow} T.$$

The domain $D = \mathbb{R} + (X, Y)\mathbb{C}[X, Y]_{(X,Y)}$ shows that, in general,

$$D \subset R \subset T, \qquad D \overset{t\text{-linked}}{\hookrightarrow} R, \qquad D \overset{t\text{-linked}}{\hookrightarrow} T \not\Rightarrow R \overset{t\text{-linked}}{\hookrightarrow} T.$$

In fact, since D' is not DW, there exists an overring T of D' which is not t-linked over D'. But T is also an overring of D, D is DW, whence T is t-linked over D.

We know that a Prüfer domain is DW, since in a Prüfer domain $t = d$, and we have already recalled that Prüfer domains are exactly the integrally closed domains such that $t = d$. We still do not know which the structure of the integrally closed DW-domains is, but we can investigate how far an integrally closed DW-domain is from being Prüfer. For example, it is easily seen that if D is DW, then D is Prüfer if and only if D is a PvMD (we recall that a PvMD is an integrally closed domain with $w = t$).

We recall that a domain D is:

- *Coherent* if the intersection of two finitely generated ideals is finitely generated;
- *Quasicoherent* if I^{-1} is finitely generated when I is a nonzero finitely generated ideal of D;
- *Finite conductor* if $aD \cap bD$ is finitely generated for each $a, b \in D$.

It is also well known that (see for instance Gabelli and Houston, 1997)

$$\text{coherent} \Rightarrow \text{quasicoherent} \Rightarrow \text{finite conductor.}$$

Using the fact that if D is quasicoherent, then D' is t-linked over D (Dobbs et al., 1989, Corollary 2.14), we have the following particular case of Corollary 3.5.

Proposition 3.6. *Let D be a quasicoherent domain such that D' is DW. Then D is DW.*

Theorem 3.7. *Let D be an integrally closed, DW domain. Then D is Prüfer if and only if D is finite conductor.*

Proof. Suppose that D is Prüfer. Then D is coherent (Gilmer, 1972, Proposition 25.4), whence it is finite conductor.

Conversely, D is an integrally closed, finite conductor domain, so D is a PvMD by Zafrullah (1978, Theorem 2). Thus D is Prüfer. □

Since a Prüfer domain is integrally closed and DW, we can reformulate the above result as follows:

$$\text{Prüfer} \Leftrightarrow DW + \text{integrally closed} + \text{finite conductor.}$$

Vasconcelos raised the following conjecture (cf. Glaz and Vasconcelos, 1984, Conjecture C3, p. 223):

"If D is a one-dimensional, coherent domain, then D' is Prüfer."

In Papick (1978) the author gives a partial answer to this question for domains D having a *very finite extension* (an extension $D \subseteq T$ is very finite if each extension S, with $D \subseteq S \subseteq T$, is of finite type as a D-module). Papick shows that if D is a one-dimensional, coherent domain such that, for each $P \in \operatorname{Spec}(D)$ with D_P not integrally closed, D_P has a very finite extension, then D' is Prüfer.

We know that one-dimensional domains are always DW. Thus, we will put ourselves in the more general context of DW, coherent domains and see what happens to their integral closure in terms of Prüfer-like conditions.

Corollary 3.8. *Let D be a domain such that D' is finitely generated as a D-module. Assume that D is DW and coherent, then D' is Prüfer if and only if D' is DW.*

Proof. The thesis directly follows from Theorem 3.7 and from the fact that D' is coherent because it is finitely generated over the coherent domain D (Harris, 1966, Corollary 1.5). □

Corollary 3.9. *Let D be a coherent, one-dimensional domain such that D' is finitely generated as a D-module. Then D' is Prüfer.*

So, in the cases in which D is DW implies D' is DW (for instance, the one-dimensional case and all the cases considered in Proposition 3.2 and Corollary 3.3) we have that

$$D \text{ is } (\text{DW} + \text{coherent}) + D \overset{\text{finite type}}{\hookrightarrow} D' \Rightarrow D' \text{ Prüfer.}$$

Remark 3.10. Different versions of Theorem 3.7 for the quasilocal case can be found in literature with hypotheses stronger than DW (and with the hypothesis of coherence, which is stronger than finite conductor). For instance:

(i) D is one-dimensional (Quentel, 1967, Corollaire 2);
(ii) D is treed (McAdam, 1972, Theorem 2);
(iii) D has a minimal overring (Papick, 1976a, Theorem 2.8).

We know that the one-dimensional and the treed domains are DW. Moreover an integrally closed, quasilocal domain having a minimal overring is DW because the maximal ideal is the radical of a principal ideal (Ayache, 2003, Theorem 1.2), hence it is a t-ideal. So, these results can all be recovered as corollaries of Theorem 3.7.

In Section 4 (Theorem 4.2) we will show that quasilocal, integrally closed Mori (in particular, Noetherian) domains which are DW are DVR's or $(M : M) \neq D$ (where M is the maximal ideal of D). But, outside the Mori case, domains with these properties may not fall in one of these two classes.

An example of a quasilocal, integrally closed domain D which is DW but it is not a valuation domain and also $(M : M) = D$ can be obtained with the following construction. In the field $\mathbb{Q}(X)$ consider any rank-one valuation domain (V, M) which is not a DVR and containing a nonzero prime number p. Put $D := V \cap \mathbb{Q}[X]$. In Loper and Tartarone (to appear, Lemmas 1.3 and 1.15) these domains are studied and it is shown that $\mathfrak{M} = M \cap \mathbb{Q}[X]$ is height-two and is the radical of (p), whence it is a t-ideal. Moreover, $\mathfrak{M}D_{\mathfrak{M}}$ is still minimal over the principal ideal generated by p and so it is a t-ideal. Then $D_{\mathfrak{M}}$ is DW. Now, $D_{\mathfrak{M}} = V_{\mathfrak{M}} \cap \mathbb{Q}[X]_{\mathfrak{M}} = V \cap \mathbb{Q}[X]_{\mathfrak{M}}$ is completely integrally closed and then $(\mathfrak{M}D_{\mathfrak{M}} : \mathfrak{M}D_{\mathfrak{M}}) = D_{\mathfrak{M}}$ (Gilmer, 1972, Theorem 34.3). Finally, $D_{\mathfrak{M}}$ is not a valuation domain, since it is two-dimensional (recall that the completely integrally closed valuation domains are one-dimensional).

Generally, we notice that the DW-property does not always localize (see Example 3.12). This is due to the fact that, in general, the w-operation on D_M (for any $M \in \mathrm{Max}(D)$) is not the restriction to D_M of the w-operation on D and so we cannot apply Proposition 2.18.

We recall that an integral domain is *v-coherent* if for each nonzero finitely generated ideal I of D, $I^v = J^{-1}$, for some finitely generated ideal J of D. Important examples of v-coherent domains are Noetherian domains, Krull domains and PvMD's.

In Mimouni (2005, Theorem 2.9) it is shown that if D is a v-coherent domain, then the DW-property localizes. But the v-coherence does not give a sharp bound for domains in which the DW-property is local. In Zafrullah (1990), the author studies when a t-prime ideal P of a domain D is such that PD_P is still a t-ideal in D_P. In this case P is called a *well-behaved* ideal and a domain D in which every t-prime ideal is well-behaved is a *well-behaved domain*. Moreover, a domain D in which all the t-maximal ideals are well-behaved is *conditionally well-behaved*. Obviously, a well-behaved domain is conditionally well-behaved. In particular, v-coherent domains are conditionally well-behaved.

Now, if D is locally DW then D is DW (Mimouni, 2005, Theorem 2.9) but the converse does not always hold (Mimouni, 2005, Example 2.10). In the next result we generalize Mimouni (2005, Theorem 2.9) replacing the v-coherence with the conditionally well-behavior of D.

Proposition 3.11. *Let D be an integral domain. Any two of the following conditions imply the third one*:

(i) *D is conditionally well-behaved*;
(ii) *D is DW*;
(iii) *D_M is DW, for each maximal ideal M of D.*

Proof. (i), (ii) $\Rightarrow$ (iii) Let M be a maximal ideal of D. Since D is DW, then M is a t-ideal. Moreover D is conditionally well-behaved, so MD_M is still a t-ideal in D_M, whence D_M is DW.

(i), (iii) $\Rightarrow$ (ii) In general, (iii) $\Rightarrow$ (ii) by Mimouni (2005, Theorem 2.9).

(ii), (iii) $\Rightarrow$ (i) Suppose that M is a t-maximal ideal of D. Since D is DW, any maximal ideal of D is a t-ideal, thus M is maximal in D. By (iii), D_M is DW, so MD_M is a t-ideal. Thus D is conditionally well-behaved. □

From the previous proposition it follows that the conditionally well-behaved domains are exactly the class of domains for which the DW-property localizes at the maximal ideals.

Example 3.12. Since there exist conditionally well-behaved domains which are not well-behaved, it may happen that D and D_M are DW-domains, for each $M \in \mathrm{Max}(D)$, but D_P is not DW for some nonmaximal, prime ideal P of D. An example of such a domain (which is given in Zafrullah, 1990, §2) is the following: take $D := V + XV_Q[X]$, where V is a valuation domain of rank >1, Q is a nonzero, nonmaximal ideal of V and X is an indeterminate over V.

Proposition 3.13. *Let D be an integral domain. Then, the following conditions are equivalent*:

(i) *D is locally DW*;
(ii) *each flat overring of D is DW.*

Proof. (i) $\Rightarrow$ (ii) Take a flat overring T of D. Let M be a maximal ideal of T. Then $T_M = D_{M \cap D}$ is DW, whence T is DW.

(ii) $\Rightarrow$ (i) The localizations of D are flat overrings. □

Since for wide classes of domains the DW-property is local (with respect to the maximal ideals), we will study some conditions which are related to the DW-property in quasilocal and, sometimes, semiquasilocal domains. Let (D, M) be a quasilocal domain, we distinguish two main cases:

- $(M : M) \neq D$;
- $(M : M) = D$.

We soon examine the first case, for which the DW-property always holds.

Proposition 3.14. *Let (D, M) be a quasilocal domain such that $(M : M) \neq D$. Then D is DW.*

Proof. Let $R := (M : M)$. Since $R \neq D$, the following pullback diagram holds:

$$\begin{array}{ccc} D & \longrightarrow & D/M \\ \downarrow & & \downarrow \\ R & \longrightarrow & R/M. \end{array}$$

Hence M is divisorial (since it is the conductor of R into D), whence M is a t-ideal and D is DW. □

Proposition 3.15. *Let D be a domain with a maximal ideal M such that $(M : M) = D$. Then M is divisorial if and only if M is invertible.*

Proof. A maximal ideal M is divisorial if and only if $M^{-1} \neq D$. With our assumption, this happens if and only if $(M : M) \neq M^{-1}$. Then $M \subsetneq MM^{-1} \subseteq D$. Whence $MM^{-1} = D$ and M is invertible. □

Corollary 3.16. *Let D be a quasisemilocal domain and M a maximal ideal of D such that $(M : M) = D$. Then M is divisorial if and only if M is principal.*

An interesting case in which $(M : M) = D$, for each maximal ideal M, is given by the completely integrally closed domains (Gilmer, 1972, Theorem 34.3). In particular, we have the following proposition.

Proposition 3.17. *Let D be a quasisemilocal completely integrally closed domain such that all the maximal ideals are finitely generated. Then D is DW if and only if each maximal ideal is principal.*

Proof. It is enough to observe that if an ideal is finitely generated, then it is a t-ideal if and only if it is divisorial. □

The hypothesis that the maximal ideals are finitely generated cannot be dropped down in the previous statement. For instance, take a one-dimensional valuation domain which is not a DVR. In this case V is DW, it is completely integrally closed and the maximal ideal is not principal.

Krull domains are completely integrally closed (Gilmer, 1972, §43). Hence we get that if D is a quasisemilocal, Krull domain, then D is DW if and only if D is a Principal Ideal Domain (PID). We notice that this result can be also easily obtained as a corollary of Mimouni (2005, Corollary 2.3), using the fact that a quasisemilocal Dedekind domain is a PID.

For the following result we recall that a nonzero ideal I in a domain D is *t-invertible* if $(II^{-1})^t = D$.

Proposition 3.18. *Let (D, M) be a quasilocal domain.*

(1) *If $(M : M) \neq D$, then D is DW*;
(2) *If $(M : M) = D$ and M is finitely generated, then D is DW if and only if M is principal*;
(3) *If $(M : M) = D$ and M is not finitely generated, then D is DW if and only if M is not t-invertible.*

Proof. (1) The thesis directly follows from Proposition 3.14.

(2) Since M is finitely generated, M is a t-ideal if and only if it is divisorial. By Corollary 3.16, M is divisorial if and only if M is principal.

(3) If M is not finitely generated, then M is not invertible. From Dobbs et al. (1989, Theorem 2.6) if D is DW, M is not t-invertible.

Conversely, if M is not t-invertible, then M is a t-ideal. In fact, if not, then $M^t = D$, and so $(MM^{-1})^t = M^t = D$. Thus M is t-invertible against the assumption. □

We notice that in Cases (1) and (2) M is divisorial, while Case (3) points to the domains in which M is a t-ideal but it is not divisorial.

Finally, we remark that, whatever domain D is, if M is not t-invertible, then M is a t-ideal and D is DW.

4. MORI AND NOETHERIAN DOMAINS

In this section, we characterize Noetherian and Mori domains which are DW.

We recall that the Noetherian domains in which $v = d$ have been widely studied starting from the sixties with the works of Bass (1962) and Heinzer (1968). Precisely, Bass gives the following characterization (Bass, 1962, Corollary 3.4):

"Let D be a Noetherian domain. Then D is divisorial if and only if its injective dimension is ≤ 1 (i.e., D is Gorenstein)."

We also observe that in the Noetherian case $v = t$, whence the above result also characterizes Noetherian domains in which $t = d$ (in the integrally closed case these are exactly the Dedekind domains).

We recall that a Mori domain is a domain in which the a.c.c. on integral divisorial ideals holds (for a wide resume see, for instance, Barucci, 2000) and a Strong Mori domain is a domain in which the a.c.c. on integral w-ideals holds (see, for instance, Wang and McCasland, 1999). The Strong Mori domains form an intermediate class of domains between the Noetherian and the Mori ones. Generally, these classes are distinct, but in the DW-case the Noetherian and the Strong Mori domain classes coincide.

Since we have shown that the one-dimensional domains are DW, we soon get the following well known result (Wang and McCasland, 1999, Corollary 1.10).

Proposition 4.1. *A one-dimensional domain is Noetherian if and only if it is Strong Mori.*

For a quasilocal, Mori domain, we have the following characterization.

Theorem 4.2. *Let (D, M) be a quasilocal, Mori domain. Then D is DW if and only if $(M : M) \neq D$ or D is a DVR.*

Proof. If $(M : M) \neq D$ or D is a DVR, then D is DW by Propositions 3.14 and 2.9.

Conversely, suppose that D is DW and that $(M : M) = D$. We will show that D is a DVR.

The maximal ideal M is a t-ideal, and since in a Mori domain $t = v$, M is divisorial. Therefore, $M^{-1} \neq D$ (otherwise we would have $M_v = (M^{-1})^{-1} = D$). By hypothesis $D = (M : M)$, whence $M^{-1} \neq (M : M)$ that is, M is not strong. By Barucci (2000, Theorem 3.4) $D = D_M$ is a DVR. □

Corollary 4.3. *Let (D, M) be a quasilocal domain such that $(M : M) = D$. Suppose that D is DW. The following conditions are equivalent:*

(i) *D is Noetherian;*
(ii) *D is Mori;*
(iii) *D is a DVR.*

The following example shows that when (D, M) is a quasilocal domain with $(M : M) \neq D$ (so D is DW), then D can be Mori but not Noetherian (and of any Krull dimension).

Example 4.4. Consider the following pullback diagram:

$$\begin{array}{ccc} D := \mathbb{Z}_p + X\mathbb{Z}_p(t)[X]_{(X)} & \longrightarrow & \mathbb{Z}_p = D/M \\ \downarrow & & \downarrow \\ B := \mathbb{Z}_p(t)[X]_{(X)} & \longrightarrow & \mathbb{Z}_p(t) = B/M \end{array}$$

where $M := X\mathbb{Z}_p(t)[X]_{(X)}$. In this case, D is a Mori domain. In fact, since B is Noetherian, M is a Mori ideal in B (Barucci, 2000, Theorem 8.1) and D is Mori from Barucci (2000, Theorem 8.2). But D is not Noetherian since the extension $\mathbb{Z}_p \hookrightarrow \mathbb{Z}_p(t)$ is not finite (Fontana, 1980, Proposition 3.1(11)). This same construction works out if we set $B := \mathbb{Z}_p(t)[X_1, \ldots, X_n]_{(X_1,\ldots,X_n)}$ and $M := (X_1, \ldots, X_n)\mathbb{Z}_p(t)[X_1, \ldots, X_n]_{(X_1,\ldots,X_n)}$.

Then $D := \mathbb{Z}_p + (X_1, \ldots, X_n)\mathbb{Z}_p(t)[X_1, \ldots, X_n]_{(X_1,\ldots,X_n)}$ is still a Mori, non Noetherian domain and $\dim(D) = n$.

More generally, we have the following characterization of Mori, DW domains.

Proposition 4.5. *Let D be a Mori domain. Then D is DW if and only if, for each maximal ideal M of D, one of the two following conditions holds*:

(1) $(M : M)D_M \neq D_M$;
(2) D_M *is a DVR.*

Proof. Since a Mori domain is conditionally well-behaved (Houston and Zafrullah, 1988, Corollary 1.8), by Proposition 3.11 D is DW if and only if D_M is DW, for each $M \in \mathrm{Max}(D)$. By Theorem 4.2, D_M is Mori if and only if $(MD_M : MD_M) \neq D_M$ or D_M is a DVR. Now, it is enough to show that the two conditions $(M : M)D_M \neq D_M$ and $(MD_M : MD_M) \neq D_M$ are equivalent. Obviously, if $(MD_M : MD_M) \neq D_M$ then $(M : M)D_M \neq D$. Conversely, if $(M : M)D_M \neq D_M$, then $(M : M) \neq D$ and $M^{-1} \neq D$, whence M is divisorial. Since D is Mori, there exists a finitely generated ideal I such that $M = I^v$. Thus $(M : M)D_M = (M : I^v)D_M = (M : I)D_M = (MD_M : ID_M) \supseteq (MD_M : MD_M)$. It follows that $(M : M)D_M = (MD_M : MD_M)$ and $(MD_M : MD_M) \neq D_M$. □

Since a Noetherian domain is Mori, we can apply the above result to this important particular case obtaining the following characterization.

Corollary 4.6. *Let D be a Noetherian domain. Then D is DW if and only if for each maximal ideal M of D one of the two following conditions holds*:

(1) $(M : M)D_M \neq D_M$;
(2) *M is invertible (then M has height one).*

Proof. It is enough to notice that condition (2) of Proposition 4.5 in the Noetherian case is equivalent to say that M is invertible (that is finitely generated and locally principal). □

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