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Flat ideals and stability in integral domains

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ABSTRACT

We introduce the concept of *quasi-stable* ideal in an integral domain D (a nonzero fractional ideal I of D is quasi-stable if it is flat in its endomorphism ring $(I : I)$) and study properties of domains in which each nonzero fractional ideal is quasi-stable. We investigate some questions about flatness that were raised by S. Glaz and W.V. Vasconcelos in their 1977 paper [17].

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Introduction

Throughout the paper D is an integral domain with quotient field K , an ideal is a fractional ideal and an integral ideal is an ideal contained in D .

The property of flatness for ideals in commutative rings has been investigated in many interesting papers. We recall some of them that inspired part of this work: J.D. Sally and W.V. Vasconcelos [34] (1975), S. Glaz and W.V. Vasconcelos [17,18] (1977, 1984), D.D. Anderson [3] (1983) and M. Zafrullah [37] (1990).

More recently many researchers have studied ideals which satisfy the following stability criterion: a nonzero ideal I of D is *stable* if I is projective in the endomorphism ring $(I : I)$ and a domain D is *stable* if each nonzero ideal of D is stable (D is *finitely stable* if each nonzero finitely generated ideal of D is stable). In particular, stable ideals and domains have been widely investigated by D.E. Rush [33] (1995), B. Olberding [29–31] (1998, 2001, 2002) and H.P. Goeters [19] (1998). Some aspects of their works on stability have been also deepened by studying properties of the class semigroup of D such

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as the Clifford regularity (cf. S. Bazzoni [7,8] (2000, 2001)). Moreover, in [25,26] S.E. Kabbaj and A. Mismouni have strengthened the notion of stable ideal (and domain) considering the so-called *strongly stable ideals*, that is nonzero ideals which are principal in their endomorphism ring (analogously, a domain D is *strongly stable* if each nonzero ideal of D is strongly stable).

In integral domains the properties of being projective and invertible for an ideal I are equivalent (analogously, free is equivalent to principal), and flatness is a natural generalization of the projective property. In [19] the condition that a nonzero ideal I is flat in $(I : I)$ is investigated in Noetherian domains and it is shown that, if D is Noetherian, this property holds for each nonzero fractional ideal of D if and only if D is stable.

In this paper we attempt to link the two concepts of flatness and stability for ideals in integral domains, by considering *quasi-stable* ideals: a nonzero ideal I is quasi-stable if it is flat in $(I : I)$. So, the quasi-stable property generalizes the stable property (instead of strengthening it as in [25]). The study of quasi-stable ideals has required a more general investigation on flatness of ideals which turned out to be useful to deepen some open problems.

Whether flat ideals of integrally closed domains are complete is a question that has been first posed in [34]. In that paper (and in the following [17]) the authors address the divisibility problem for flat ideals, that is, the problem of deciding when an element belongs to a flat ideal. One of the main tools in this study is what they called “the divisibility lemma”, which is, in modern language, the fact that a flat ideal is a w -ideal. In the introduction of [17], the authors say that the last section of that article “contains a number of unresolved questions where the elusive completeness of flat ideals plays a significant role” and they add later in the paper that “unfortunately other than the few cases of [34], not much seems known” (the cases are those of Krull domains, GCD-domains and integrally closed coherent domains, cf. [34, Example 1.5]). In Section 1 we improve the divisibility lemma (Theorem 1.4), by showing that a flat ideal is not only a w -ideal, but it is in fact a t -ideal, and obtain, by using some well-known properties of star-operations, the completeness of flat ideals in integrally closed domains.

Another question considered in [34] and [17] is related to the characterization of domains in which flat ideals are finitely generated (and so, invertible). For example, in [34, Theorem 3.1] it is shown that a flat ideal of a polynomial ring with finitely generated content is invertible. It is also observed that flat ideals in Krull domains are invertible. In [17, §3] it is conjectured that faithfully flat ideals in H -domains are invertible (an H -domain is a domain in which every t -maximal ideal is divisorial). We show that this is not true, by giving a counterexample (Example 1.10). On the other side, we show that the t -finite character on D suffices to have that all faithfully flat ideals are invertible (Proposition 1.13). This result may be related to the Bazzoni’s conjecture [5], recently proven in [23] and in [20], which states that all locally invertible (i.e., faithfully flat) ideals of a Prüfer domain are invertible if and only if the domain has the (t) -finite character on maximal ideals.

In Section 2, with the necessary assumption of the t -finite character, we characterize stable domains as the domains in which each ideal is faithfully flat in its endomorphism ring (Proposition 2.1). So, it seems natural to define a new class of domains, the *quasi-stable domains*, that is, the domains such that each nonzero ideal is flat in its endomorphism ring. We show that this class is strictly larger than the class of stable domains (this is easy to see) and, with an elaborate example, that it is smaller than the class of finitely stable domains, even if these two classes coincide for Noetherian and integrally closed domains.

In Section 3, we study overrings and localizations of quasi-stable domains and show that they are still quasi-stable in some significant cases.

1. Flat ideals and t -ideals

We recall some basic terminology and notions about divisorial ideals, t -ideals and w -ideals. Given a domain D with quotient field K , we put $\mathfrak{F}(D)$ to be the set of nonzero D -modules contained in K , $\mathbf{f}(D)$ the set of nonzero finitely generated D -modules contained in K and $\mathbf{F}(D)$ the set of nonzero fractional ideals of D .

If I is a nonzero ideal of D , then:

- the *divisorial closure* of I is the ideal $I_v := (D : (D : I))$, where $(D : H) := H^{-1} := \{x \in K \mid xH \subseteq D\}$, for each $H \in \mathbf{F}(D)$;
- the *t -closure* of I is the ideal $I_t := \bigcup_{J \in \mathbf{f}(D), J \subseteq I} J_v$;
- the *w -closure* of I is the ideal $I_w := \bigcup_{J \in \mathbf{f}(D), J_v = D} (I : J)$.

An ideal $I \in \mathbf{F}(D)$ is *divisorial* (respectively, a *t -ideal* or a *w -ideal*) if $I = I_v$ (respectively, $I = I_t$ or $I = I_w$). For each $I \in \mathbf{F}(D)$, the following inclusions hold: $I \subseteq I_w \subseteq I_t \subseteq I_v$.

An ideal I is *t -finite* if there exists a finitely generated ideal $J \subseteq I$ such that $J_t = I_t$.

The v -, t - and w -operations are particular *star-operations* (see, for instance, [28,14]). The t -operation is a *star-operation of finite type*, that is, for each $H \in \mathbf{F}(D)$:

$$H_t := \bigcup \{F_t \mid F \subseteq H, F \in \mathbf{f}(D)\}.$$

Moreover, t is maximal among the star-operations of finite type on D that is, if \star is a finite-type star-operation on D , then $\star \leq t$ (i.e., $H_\star \subseteq H_t$, for each $H \in \mathbf{F}(D)$).

An ideal of a domain D is flat if it is flat as a D -module. A useful characterization of flat ideals in integral domains is the following [3, Theorem 2]:

Proposition 1.1. *Let D be an integral domain. An ideal I of D is flat if and only if $(A \cap B)I = AI \cap BI$ for all $A, B \in \mathbf{F}(D)$.*

Being projective, invertible ideals are flat. We give a short proof of this fact, by using the previous characterization. Note that it is always true that, if A, B and C are (fractional) ideals of D , then $C(A \cap B) \subseteq CA \cap CB$. So, let I be invertible and A and B ideals of D . Then:

$$IA \cap IB = II^{-1}(IA \cap IB) \subseteq I(I^{-1}IA \cap I^{-1}IB) = I(A \cap B).$$

Thus I is flat.

Note that flat ideals are not always invertible. For example, we recall that Prüfer domains are exactly the domains in which each ideal is flat ([16, Theorem 25.2] and Proposition 1.1). So, in a non-Dedekind Prüfer domain, each non-finitely generated ideal is flat but not invertible (we can take $D := \text{Int}(\mathbb{Z}) := \{f(X) \in \mathbb{Q}[X] \mid f(\mathbb{Z}) \subseteq \mathbb{Z}\}$ [10, §6]).

On the contrary, it is well known that even in the more general context of rings with zero divisors finitely generated ideals are flat if and only if they are projective. So, in a domain, finitely generated flat ideals are invertible. More precisely we have the following [37, Proposition 1]:

Proposition 1.2. *Let D be an integral domain and I a t -finite ideal of D . Then I is flat if and only if it is invertible.*

A consequence of this fact is that in Krull and Noetherian domains (and more generally in Mori domains), the flat ideals are exactly the invertible ideals [37, Corollary 4].

It is known that flat ideals are w -ideals (or semidivisorial ideals, in the language of Glaz and Vasconcelos [17, Corollary 2.3]). We can show that flat ideals are in fact t -ideals. We will use the following lemma.

Lemma 1.3. *Let D be an integral domain, J a nonzero finitely generated ideal of D . If I is a flat ideal of D , then $(I : J) = IJ^{-1}$.*

Proof. Let $J = (a_1, a_2, \dots, a_n)$. Then, by the flatness of I , we have that:

$$(I : J) = (a_1^{-1}I \cap a_2^{-1}I \cap \dots \cap a_n^{-1}I) = I(a_1^{-1}D \cap a_2^{-1}D \cap \dots \cap a_n^{-1}D) = IJ^{-1}. \quad \square$$

Theorem 1.4. Let D be an integral domain and I be a nonzero ideal of D . If I is flat then I is a t -ideal.

Proof. Let J be a nonzero finitely generated ideal. Then, since I is flat, $(I : J) = IJ^{-1}$ (by Lemma 1.3). Now, $J^{-1} = (J_v)^{-1}$, hence:

$$(I : J) = IJ^{-1} = I(J_v)^{-1} = I \bigcap_{x \in J_v} \frac{1}{x} D \subseteq \bigcap_{x \in J_v} \frac{1}{x} I = (I : J_v) \subseteq (I : J).$$

Thus $(I : J) = (I : J_v)$. If $J \subseteq I$, then $1 \in (I : J)$. So, $1 \in (I : J_v)$, that is $J_v \subseteq I$. Hence $I = I_t$. \square

Remark 1.5. (1) A divisorial ideal (and so a t -ideal) is not always flat. For instance, take a non-integrally closed domain D in which each ideal is divisorial (e.g., a pseudo-valuation, non-valuation, domain such that the associated valuation domain is two-generated as a D -module [22, Corollary 1.8]). Then D has, at least, a nonzero ideal which is not flat, otherwise D would be a valuation domain.

(2) Note that prime flat t -ideals are *well behaved* in the sense of Zafrullah (a prime t -ideal P of D is well behaved if PD_P is a t -ideal in D_P [36]). This follows from the fact that, for ideals, flat implies locally flat, and flat implies t -ideal. So, a prime t -ideal which is not well behaved is not flat.

(3) In [37, Proposition 10], M. Zafrullah has shown that the integral domains in which each t -ideal is flat are precisely the generalized GCD domains (G-GCD domains) defined in [1], that is the domains in which each t -finite t -ideal is invertible.

An immediate corollary of Theorem 1.4 is that in the statement of Lemma 1.3 J can be taken t -finite instead of finitely generated.

Corollary 1.6. Let D be an integral domain and J be a nonzero t -finite ideal of D . If I is a flat ideal of D then $(I : J) = IJ^{-1}$.

Proof. Let H be a finitely generated ideal of D such that $H_t = J_t$. By Theorem 1.4 and Lemma 1.3 it follows that:

$$(I : J) = (I_t : J_t) = (I_t : H_t) = (I : H) = IH^{-1} = IJ^{-1}. \quad \square$$

We recall that for each $I \in \mathbf{F}(D)$, the b -closure of I is defined as follows:

$$I^b := \bigcap IV_\alpha,$$

where the intersection is taken over all valuation overrings V_α of D . An ideal I is called *complete* if it is a b -ideal, that is, if $I^b = I$ [16, §24]. As shown in [38, Appendix 4, Theorem 1], the b -closure of an ideal of D coincides with the integral closure of I in K . By [38, Appendix 4, Theorem 1] and the definition of integral dependence and integral closure it follows easily that, if D is integrally closed, the b -operation is a star-operation and it is of finite type. If D is not integrally closed, the b -closure can be still defined as above for each $I \in \mathbf{F}(D)$, but in this case it is not a star-operation; it is actually a semistar-operation, which is a generalization of star-operation, that we don't need to discuss in this context.

In [17, Conjecture, p. 16], the authors conjecture that a flat ideal of an integrally closed domain is complete.

Theorem 1.7. (Cf. [17, Conjecture, p. 16].) *Every flat ideal of an integrally closed domain is complete.*

Proof. Let D be an integrally closed domain. As remarked above, the b -operation on D is a star-operation of finite type, so $b \leq t$, that is, $I^b \subseteq I_t$, for each $I \in \mathbf{F}(D)$. Thus t -ideals are complete. From Theorem 1.4, flat ideals are t -ideals, whence they are complete. \square

In [17, p. 16] the authors prove that *if A is an integrally closed domain of characteristic 2, then an idempotent flat ideal of A is a radical ideal.* By using Theorem 1.7, we can prove this result in any characteristic.

Proposition 1.8. *Let D be an integrally closed domain. Then, an idempotent flat ideal of D is a radical ideal.*

Proof. Let I be a flat, idempotent ideal of D . By hypothesis, $D = \bigcap_{\alpha \in A} V_\alpha$, where $\{V_\alpha\}_{\alpha \in A}$ are all the valuation overrings of D . Then, for each $\alpha \in A$, IV_α is idempotent and so prime [16, Theorem 17.1]. Let $IV_\alpha = P_\alpha$. Since I is flat, then I is complete (Theorem 1.7) and so $I = \bigcap_{\alpha \in A} IV_\alpha = \bigcap_{\alpha \in A} P_\alpha = \bigcap_{\alpha \in A} (P_\alpha \cap D)$ is an intersection of prime ideals. Thus I is a radical ideal. \square

Remark 1.9. Note that if all ideals of a domain D are complete then D is a Prüfer domain [16, Theorem 24.7] and so all ideals are flat [16, Theorem 25.2 (c)]. In general, it is not always true that complete ideals are flat. For instance, a prime ideal P of an integrally closed domain D is always complete since there always exists a valuation overring of D centered on P [16, Theorem 19.6]. But, obviously, P is not always a t -ideal thus, in particular, it is not always flat. Such an example is given by a height-2 prime ideal of $\mathbb{Z}[X]$. In fact, since $\mathbb{Z}[X]$ is a Krull domain, it is well known that the only prime t -ideals are the height-one primes.

We recall that a domain D is an H -domain if for each ideal I of D such that $I^{-1} = D$, there exists a finitely generated ideal $J \subseteq I$ such that $J^{-1} = D$. In [24, Proposition 2.4] it is shown that this is equivalent to the fact that each t -maximal ideal of D (i.e., an ideal which is maximal in the set of t -ideals of D) is divisorial.

In [17, Proposition 1.1] it is shown that an ideal I of a domain D is faithfully flat (as a D -module) if and only if it is flat and locally finitely generated. This is equivalent to saying that I is faithfully flat if and only if it is locally invertible [2, Theorem 8].

A second conjecture stated in [17, p. 9] is the following:

Conjecture 2. (Cf. [17, Conjecture, p. 9].) *A faithfully flat ideal in an H -domain is finitely generated.*

Now, we give a counterexample showing that this conjecture is false.

Example 1.10. We recall that generalized Dedekind domains (see, for instance, [15,32]) are examples of H -domains (since their prime ideals are divisorial [15, Theorem 15]). Now, consider the domain $D := \mathbb{Z} + X\mathbb{Q}[[X]]$. In [15, Example 2] it is shown that D is generalized Dedekind. Let I be the ideal of D generated by the set $\{\frac{1}{p}X \mid p \in \mathbb{Z}\}$. It is easy to check that I is locally principal. Moreover, in [15] it is also shown that I is not divisorial. Then I is not finitely generated, otherwise it would be invertible and so divisorial.

Remark 1.11. Conjecture 2 may be refuted also by using the following argument. R. Gilmer [16, Lemma 37.3] has shown that:

Lemma 1.12. *If D is a Prüfer domain with the finite character (i.e., each nonzero element of D is contained in finitely many maximal ideals), then every locally principal ideal (i.e., faithfully flat ideal) of D is invertible.*

In [5, p. 630] S. Bazzoni conjectured that:

“Let D be a Prüfer domain. Then every locally principal ideal of D is invertible if and only if D has the finite character.”

and proved this conjecture for some particular Prüfer domains [5, Theorem 4.3]. Recently this conjecture has been proven by W.C. Holland, J. Martinez, W.Wm. McGovern, M. Tesemma [23] and, independently, by F. Halter-Koch [20].

In Example 1.10 we have recalled that generalized Dedekind domains are H-domains. Now, if Conjecture 2 were true, a generalized Dedekind domain D would be a Prüfer domain in which each locally principal ideal is invertible. Hence D would have the finite character. But the domain $\mathbb{Z} + X\mathbb{Q}[[X]]$ considered in Example 1.10 is generalized Dedekind without the finite character (the element X is contained in infinitely many maximal ideals).

In [17] the authors have shown that to prove Conjecture 2 would be enough to get that each faithfully flat ideal in an H-domain is divisorial. In Example 1.10 we have seen that this is not always true, but we have shown that (faithfully) flat ideals are t -ideals (Theorem 1.4). Now, recall that H-domains are exactly the domains in which the t -maximal ideals are all divisorial. Note that if we strengthen this condition considering domains in which all the t -ideals are divisorial (*TV-domains*, cf. [24]), then for this class of domains Conjecture 2 is true, since, in this case, flat ideals, being t -ideals, are divisorial. In fact, we prove something more:

Proposition 1.13. *Let D be a domain with the t -finite character (i.e., each proper t -ideal is contained in finitely many t -maximal ideals). Then each faithfully flat ideal in D is invertible.*

Proof. If $I \in \mathbf{F}(D)$ is faithfully flat, then I is locally principal and, in particular, I is t -locally principal (i.e., ID_P is principal for each $P \in t\text{-max}(D)$). The t -finite character of D implies that I is t -finite. Then, by Proposition 1.2, I is invertible. \square

Since TV-domains have the t -finite character [24, Theorem 1.3], we obtain the following:

Corollary 1.14. *Let D be a TV-domain. Then each faithfully flat ideal in D is invertible.*

Remark 1.15. Given an integral domain D consider the two following conditions:

- (a) D has the t -finite character;
- (b) each faithfully flat ideal in D is invertible.

Proposition 1.13 proves that (a) \Rightarrow (b) for any domain D .

We notice that for Prüfer domains (b) \Rightarrow (a) (in this case $t = d$ and (a) is the hypothesis of finite character on D). This is exactly the content of Bazzoni’s conjecture.

Moreover, if D is a Noetherian domain, it is well known that each faithfully flat ideal in D is invertible (see also Proposition 1.2). A Noetherian domain does not necessarily have the finite character, but it does have the t -finite character. So, also in this case we have that (b) \Rightarrow (a).

What we observed for these two relevant classes of domains (the Prüfer and the Noetherian ones) suggests the following question:

Question 1.16. If each faithfully flat ideal of D is invertible, does D have the t -finite character?

So far, we are not able to answer to this questions but the considerations above suggest to investigate in this direction and try to generalize Bazzoni’s conjecture to a class of domains larger than the one of Prüfer domains.

2. Quasi-stable domains

We recall from the Introduction that a nonzero ideal I of D is *stable* if I is projective in the endomorphism ring $(I : I)$ and that D is a *stable domain* if each nonzero ideal of D is stable. Moreover, an integral domain D is *finitely stable* if each nonzero finitely generated ideal of D is stable.

Proposition 1.13 suggests the following characterization of stable domains with the t -finite character.

Proposition 2.1. *An integral domain D with the t -finite character is stable if and only if each nonzero ideal I of D is faithfully flat in $(I : I)$.*

Proof. If D is stable, then each nonzero ideal I of D is invertible in $(I : I)$, and so I is faithfully flat in $(I : I)$ (and this is true even without assuming the t -finite character).

For the converse, first note that if each nonzero ideal I of D is faithfully flat in $(I : I)$ then, in particular, each finitely generated ideal I is invertible in $(I : I)$. Thus D is finitely stable. By [33, Proposition 2.1], finitely stable domains have Prüfer integral closure, whence all maximal ideals of D are t -ideals by [12, Lemma 2.1 and Theorem 2.4]. Thus, the t -finite character on D is, in fact, the finite character and, by [31, Lemma 3.4], all overrings of D have the finite character. By hypothesis, if $I \in \mathbf{F}(D)$, I is faithfully flat in $(I : I)$ (which has the finite character). So I is invertible by Proposition 1.13 and D is stable. \square

Remark 2.2. If D does not have the t -finite character, Proposition 2.1 does not hold. In fact, take an almost Dedekind domain D which is not Dedekind [16, Example 42.6 and Remark 42.7]. In this case $t = d$ (D is Prüfer) and D does not have the t -finite character. Each ideal of D is locally principal and so it is faithfully flat in D . Moreover, D is the endomorphism ring of each of its ideals, since it is completely integrally closed, but D has, at least, a nonzero ideal which is not invertible and so D is not stable.

After considering the faithfully flat condition on ideals, it seems natural to investigate in which domains each nonzero ideal is flat in its endomorphism ring and compare this new class of domains with stable and finitely stable domains.

Definition 2.3. We say that a nonzero ideal I of a domain D is *quasi-stable* if I is flat as an ideal of $(I : I)$ and that a domain D is *quasi-stable* if each nonzero ideal of D is quasi-stable.

Proposition 2.4. *The following conditions are equivalent for an integral domain D :*

- (i) D is finitely stable.
- (ii) Each nonzero finitely generated ideal of D is quasi-stable.
- (iii) For each nonzero finitely generated ideal I of D , I is a t -ideal of $(I : I)$ and $((I : I) : I)$ is a finitely generated ideal of $(I : I)$.

Proof. (i) \Leftrightarrow (ii) and (ii) \Rightarrow (iii) are a straightforward consequence of the fact that finitely generated flat ideals are invertible.

(iii) \Rightarrow (i) follows by applying exactly the same argument used in the proof of [31, Theorem 3.5, (ii) \Rightarrow (i)]. \square

So, in particular, the Noetherian quasi-stable domains are exactly the Noetherian stable domains (cf. [19, Theorem 11]).

Note that since stable ideals are quasi-stable (invertible ideals are flat), stable domains are quasi-stable. Moreover, it is an easy consequence of Proposition 2.4 that quasi-stable domains are finitely stable.

In Example 2.9, we will show that there exists an integral domain R that satisfies condition (iii) of Proposition 2.4, but which is not quasi-stable. Thus we pose the following question:

Question 2.5. Are the finitely stable domains the domains in which each ideal (or each finitely generated ideal) is a t -ideal in $(I : I)$?

This question is also suggested by the following fact. Olberding in [31, Theorem 3.5] has shown that a domain D is stable if and only if each nonzero ideal I of D is divisorial in its endomorphism ring $(I : I)$. Moreover, the t -operation is the finite-type operation associated to the v -operation and the finitely stable domains are the finite-type version of stable domains. Thus a positive answer to the question above would give a finite-type interpretation of Olberding's result.

Examples 2.6. (1) *A quasi-stable domain that is not stable.*

Each Prüfer domain is quasi-stable, because each ideal of a Prüfer domain is flat and overrings of Prüfer domains are Prüfer. Since stable domains have the finite character [31, Theorem 3.3], it is enough to take a Prüfer domain without the finite character (e.g., an almost Dedekind domain which is not Dedekind) to get an example of a quasi-stable domain which is not stable.

Note also that the finite character on D is not sufficient to get that a quasi-stable domain is stable. Again, a Prüfer domain of finite character which is not strongly discrete (i.e., it has at least a prime ideal that is idempotent) is quasi-stable but not stable [29, Theorem 4.6].

(2) *A quasi-stable non-Prüfer domain that is not-stable.*

Consider a pseudo-valuation domain D that is not a valuation domain with maximal ideal M and associated valuation domain $M^{-1} = (M : M) = V$ and assume that V is 2-generated as a D -module. In this case $v = t = d$ on D ([22, Corollary 1.8] and [24, Proposition 4.3]). So, each ideal of D is principal or it is a common ideal of D and V [21, Proposition 2.14]. If I is principal in D , then $(I : I) = D$ and I is flat in D . So I is quasi-stable. If I is a common ideal of D and V , then $(I : I) \supseteq V$ is a valuation domain and so I is flat in $(I : I)$. Thus D is quasi-stable.

If we take M non-principal in V , then M is not invertible in $(M : M) = V$ and D is not stable.

As we have seen, it is easy to find examples of quasi-stable domains which are not stable, even in the case of integrally closed domains with finite character. On the contrary, it seems that quasi-stable domains are very close to finitely stable domains. We have already mentioned that these two classes of domains (quasi-stable and finitely stable) do coincide in the Noetherian case. The next result shows that they coincide also in the other classical case of integrally closed domains.

Proposition 2.7. *Let D be an integrally closed domain. The following conditions are equivalent:*

- (i) D is a quasi-stable domain.
- (ii) D is a finitely stable domain.
- (iii) D is a Prüfer domain.

Proof. (i) \Rightarrow (ii) follows from Proposition 2.4.
 (ii) \Rightarrow (iii) follows from [33, Proposition 2.1].
 (iii) \Rightarrow (i) is obvious. \square

Despite of the previous examples, in general finitely stable domains are not necessarily quasi-stable. The follow-up of this section is devoted exclusively to the construction of an example of a finitely stable domain which is not quasi-stable.

Example 2.8 (*Example of a domain that is finitely stable but not quasi-stable*). Let \mathbb{F}_2 be the field with 2 elements and t be an indeterminate over \mathbb{F}_2 . Let (V, M) be a DVR with residue field $\mathbb{F}_2(t)$: for instance, take $(V, M) := (\mathbb{F}_2(t)[[X]], X\mathbb{F}_2(t)[[X]])$, and consider the 2-degree field extension $\mathbb{F}_2(t^2) \subsetneq \mathbb{F}_2(t)$. Let $A := \mathbb{F}_2[t^2]_Q$, where Q is a nonzero prime ideal of $\mathbb{F}_2[t^2]$ which does not contain t^2 . Then A is a DVR with quotient field $\mathbb{F}_2(t^2)$. Consider the following pullback diagram:

$$\begin{array}{ccc}
 R := \varphi^{-1}(A) & \longrightarrow & A = R/M \\
 \downarrow & & \downarrow \\
 D := \varphi^{-1}(\mathbb{F}_2(t^2)) & \longrightarrow & \mathbb{F}_2(t^2) = D/M \\
 \downarrow & & \downarrow \\
 V & \xrightarrow{\varphi} & \mathbb{F}_2(t) = V/M
 \end{array}$$

where the horizontal arrows are projections and the vertical arrows are injections. Now, D is a Noetherian, pseudo-valuation domain and since $[\mathbb{F}_2(t) : \mathbb{F}_2(t^2)] = 2$, D is totally divisorial by [22, Corollary 1.8] and [24, Proposition 4.3] (i.e., each ideal of D is divisorial and the same holds for each overring of D). Then D is stable by [31, Theorem 2.5], whence it is finitely stable.

Let \bar{R} denote the integral closure of R . Since $R \subseteq \bar{R} \subseteq V$, M is a common ideal of R , \bar{R} and V . Then $A \subseteq \bar{R}/M \subseteq \mathbb{F}_2(t)$ and \bar{R}/M is the integral closure of A in $\mathbb{F}_2(t)$ [9, Lemme 2], that we denote, as usual, by $\bar{A}^{\mathbb{F}_2(t)}$. It follows immediately that $R \neq \bar{R}$ because $t \in \bar{A}^{\mathbb{F}_2(t)} \setminus A$ (whence, the quotient field of $\bar{A}^{\mathbb{F}_2(t)}$ is $\mathbb{F}_2(t)$). It is well known that $\bar{A}^{\mathbb{F}_2(t)}$ is the intersection of the valuation domains extending A in $\mathbb{F}_2(t)$ [16, Theorem 20.1] and, by [16, Corollary 20.3], the number of these extensions is, at most, the separable degree of the field extension $\mathbb{F}_2(t^2) \subseteq \mathbb{F}_2(t)$, which is 1. Hence, $\bar{A}^{\mathbb{F}_2(t)}$ is simply a DVR [16, Theorem 19.16 (d)]. Thus, $\bar{R} = \varphi^{-1}(\bar{A}^{\mathbb{F}_2(t)})$ is a two-dimensional valuation domain in which M is the height-one prime ideal [13, Theorem 2.4]. Moreover, the maximal ideal of \bar{R} is principal since \bar{R}/M is a DVR, and $\bar{R}_M = V$, which is a DVR, whence the nonzero prime ideals of \bar{R} are not idempotent and \bar{R} is totally divisorial [6, Proposition 7.6].

By [30, Proposition 3.6] R is finitely stable with principal maximal ideal N . By general properties of pullback constructions, R is 2-dimensional with ordered spectrum $(0) \subset M \subset N$, and $R_M = D$. Since D is 1-dimensional and R is 2-dimensional, \bar{R} does not contain D . Moreover, D does not contain \bar{R} because D is not Prüfer and \bar{R} does. So \bar{R} and D are not comparable.

Claim. Each ring between R and V is comparable with D or \bar{R} . First notice that M is a common ideal of all rings between R and V . Let B be such a ring and suppose that B is not comparable with D . Then $B/M \not\subseteq \mathbb{F}_2(t^2)$ (since $D = \varphi^{-1}(\mathbb{F}_2(t^2))$). But $A \subseteq B/M$ (because $R \subseteq B$), so $\bar{A}^{\mathbb{F}_2(t)} \subseteq B/M^{\mathbb{F}_2(t)}$. As $\bar{A}^{\mathbb{F}_2(t)}$ being a DVR, it follows that $B/M^{\mathbb{F}_2(t)} = \bar{A}^{\mathbb{F}_2(t)}$ or $B/M^{\mathbb{F}_2(t)} = \mathbb{F}_2(t)$. In the first case, we have that $B/M \subseteq \bar{A}^{\mathbb{F}_2(t)}$ and so $B \subseteq \bar{R}$ (recall that $\bar{R} = \varphi^{-1}(\bar{A}^{\mathbb{F}_2(t)})$). The second case occurs if and only if $B/M = \mathbb{F}_2(t)$ and so $B = V$, which contains \bar{R} .

By [31, Theorem 4.11], R is not stable because $R_M = D$ is not a valuation domain. Hence there exists a nonzero ideal in R which is not divisorial in $(I : I)$ [31, Theorem 3.5]. Our aim is to show that this specific ideal I is not flat in $(I : I)$ and so R is not quasi-stable.

If I is finitely generated, then I is stable since R is finitely stable and so I is divisorial in $(I : I)$.

Then we can suppose that I is not finitely generated and we distinguish the following cases:

- (a) $(I : I) = R$;
- (b) $(I : I) \neq R$ and $(I : I)$ is comparable with D ;
- (c) $(I : I) \neq R$ and $(I : I)$ is comparable with \bar{R} .

(a) If $(I : I) = R$ and I is flat in R , then I is principal or $I = IN$ by [34, Lemma 2.1]. We are supposing that I is not finitely generated, so $I = IN$. But $N = \pi R$ is principal and $I = I\pi$ implies that $\pi, \pi^{-1} \in (I : I) = R$, which is impossible. So in this case I is not flat in $(I : I)$ and R is not quasi-stable.

(b) If $(I : I) \neq R$ and $(I : I)$ is comparable with D , then $D \subseteq (I : I)$ because between R and D there are no domains, since there are no domains between A and $\mathbb{F}_2(t^2)$ (because A is a DVR). But D is totally divisorial, whence I would be divisorial in $(I : I)$ against the assumption. Thus, this case cannot occur.

(c) If $(I : I) \neq R$ and $(I : I)$ is comparable with \bar{R} , then $\bar{R} \subseteq (I : I)$ or $(I : I) \subsetneq \bar{R}$. In the first case, since \bar{R} is totally divisorial, I would be divisorial in $(I : I)$, against the assumption. So, we can assume that $(I : I) \subsetneq \bar{R}$. Then $A \subsetneq (I : I)/M \subsetneq \bar{A}^{\mathbb{F}_2(t)}$. So $(I : I)/M$ is local (since its integral closure is $\bar{A}^{\mathbb{F}_2(t)}$), it is Noetherian (by Krull–Akizuki Theorem) and it is not a PID. In fact, $(I : I)/M$ is not integrally closed (since it is strictly in between A and $\bar{A}^{\mathbb{F}_2(t)}$). It follows that $(I : I)$ is two-dimensional, with prime spectrum $(0) \subsetneq M \subsetneq \mathcal{M}$ and \mathcal{M} is not principal (since \mathcal{M}/M is not principal). If I is flat in $(I : I)$, then I is principal or $I\mathcal{M} = I$ (again by [34, Lemma 2.1]). Since I is supposed to be not divisorial in $(I : I)$, we have that $I\mathcal{M} = I$. Thus, $(\mathcal{M} : \mathcal{M}) \subseteq (I\mathcal{M} : I\mathcal{M}) = (I : I)$, and so $(\mathcal{M} : \mathcal{M}) = (I : I)$. But \mathcal{M} is not principal and $\mathcal{M}^2 \neq \mathcal{M}$, since \mathcal{M}/M is not idempotent, as being \mathcal{M}/M finitely generated. Then \mathcal{M} is not flat in $(I : I) = (\mathcal{M} : \mathcal{M})$. We finally notice that $(I : I)$ is an overring of R , which is finitely stable, whence $(I : I)$ is finitely stable. Thus, in this case, $(I : I)$ is an example of finitely stable domain, which is not quasi-stable.

We remark that, from a result that we will prove in the next section (Corollary 3.8), we also have that $(I : I)$ non-quasi-stable implies that R is not quasi-stable too.

Example 2.9. Consider the domain R constructed in the example above. We have seen that R is finitely stable but not quasi-stable. We now show that each nonzero ideal I of R is a t -ideal in $(I : I)$.

Without loss of generality we can consider only integral ideals.

By construction, each integral ideal I of R is comparable with M .

Suppose that $M \subsetneq I$, then $I = \pi^s R$ is principal, thus it is a t -ideal (recall that the maximal ideal of R is $N = \pi R$ and R/M is a DVR).

Conversely, let $I \subseteq M$. We consider two sub-cases:

(a) The domain $(I : I)$ is comparable with D .

If $D \subseteq (I : I)$, then $(I : I)$ is a divisorial domain (since D is totally divisorial) and so each ideal of $(I : I)$ is a t -ideal.

If $R \subseteq (I : I) \subsetneq D$, then $(I : I) = R$ and I is M -primary in R . By [4, Proposition 4.8], $IR_M \cap R = I$. But $R_M = D$, ID is a t -ideal in D , so I is a t -ideal in R .

(b) The domain $(I : I)$ is comparable with \bar{R} .

If $\bar{R} \subseteq (I : I)$, then $(I : I)$ is a Prüfer domain and so each ideal is a t -ideal.

If $R \subsetneq (I : I) \subsetneq \bar{R}$, then the quotient field of $(I : I)/M$ is $\mathbb{Z}_2(t)$. Then $(I : I)_M = V$, I is M -primary in $(I : I)$, IV is a t -ideal and so I is a t -ideal by the same argument used above.

3. Overrings of quasi-stable domains

It is known that overrings of stable domains are stable and overrings of finitely stable domains are finitely stable [31, Theorem 5.1 and Lemma 2.4]. In this section we study the quasi-stability for overrings of quasi-stable domains. We are able to prove that overrings of quasi-stable domains are still quasi-stable for some relevant classes of overrings (a general result is given in Corollary 3.7).

The first result of this section is a generalization of the flatness criterion for ideals in integral domains recalled in Proposition 1.1.

Proposition 3.1. *Let D be an integral domain and I be a nonzero ideal of D . Then I is flat over D if and only if $I(A \cap B) = IA \cap IB$, for all A, B D -submodules of K .*

Proof. The “if” part is already shown in Proposition 1.1.

As regards the “only if” part, the proof easily follows from the fact that if I is a flat D -module and A, B are D -submodules of K , then $I \otimes_D (A \cap B) = (I \otimes_D A) \cap (I \otimes_D B)$ [27, Theorem 7.4]. \square

Proposition 3.2. *Let D be an integral domain and I be a nonzero ideal of D . Let T be an overring of D . If I is a flat ideal of D then IT is a flat ideal of T .*

Proof. It is enough to observe that the T -submodules of K are also D -submodules of K and apply Proposition 3.1. \square

Corollary 3.3. *Let D be an integral domain and I be a nonzero ideal of D .*

- (a) *If I is flat, then I is quasi-stable.*
- (b) *If I is a flat ideal of D , then I is a t -ideal of $(I : I)$.*

Proof. (a) It is immediate from Proposition 3.2, since $(I : I)$ is an overring of D and $I = I(I : I)$.
 (b) It follows from (a) and Theorem 1.4. \square

We recall the following result due to D.E. Rush [33, Proposition 2.1].

Proposition 3.4. *Let D be a finitely stable domain. Then the integral closure \bar{D} of D is a Prüfer domain.*

Since quasi-stable domains are finitely stable we have the following corollary:

Corollary 3.5. *The integral closure of a quasi-stable domain is a Prüfer domain and so it is quasi-stable.*

Proposition 3.6. *Let D be an integral domain and T be an overring of D . If I is a quasi-stable ideal of D , then IT is a quasi-stable ideal of T .*

Proof. Since I is flat in $(I : I)$, then $IT = I(I : I)T$ is flat in $(I : I)T$, by Proposition 3.2. Now, $(I : I)T \subseteq (IT : IT)$, so applying again Proposition 3.2, we obtain that IT is a flat ideal of $(IT : IT)$. \square

As it is stated in the next result, a case in which the quasi-stability transfers to overrings is when we have a ring extension $D \hookrightarrow T$ such that map

$$\Phi_D^T : \mathbf{F}(D) \rightarrow \mathbf{F}(T), \quad I \mapsto IT$$

is surjective that is, when each ideal of T is an extension of an ideal of D (we remark that this includes also the case when an integral ideal of T is an extension of a fractional ideal of D).

Corollary 3.7. *Let D be an integral domain and let T be an overring of D such that Φ_D^T is surjective. Then, if D is quasi-stable, T is quasi-stable.*

Proof. It is an immediate consequence of Proposition 3.6. \square

Interesting classes of overrings of a domain D which satisfy the condition of Corollary 3.7 are studied in [35] and we list them as follows:

- T is an overring of D such that $(D : T) \neq 0$ (a particular case is when $T = (I : I)$);
- T is a flat overring of D (i.e., T is flat as a D -module);
- T is a Noetherian overring of D ;
- T is well centered on D (i.e., for all $t \in T$ there exists $u \in U(D)$ such that $ut \in D$);
- T is any overring of a domain D such that \bar{D} is Prüfer and it is a (fractional) ideal of D .

Recalling that if D is quasi-stable then D is finitely stable and so its integral closure is Prüfer, from the last point of the list above we get the following:

Corollary 3.8. *Let D be an integral domain such that $(D : \bar{D}) \neq (0)$. If D is quasi-stable, then every overring of D is quasi-stable.*

A domain D is called *conductive* if $(D : T) \neq (0)$ for all overrings of D .

Corollary 3.9. *An overring of a conducive quasi-stable domain is quasi-stable.*

Note that there exist quasi-stable domains which are not conducive (for example, not all Prüfer domains are conducive).

The study of stability and finite stability can be reduced to the local case, since a domain is stable if and only if it is locally stable and it has the finite character [31, Theorem 3.3], and it is finitely stable if and only if it is locally finitely stable. We approach this question in the case of quasi-stable domains.

Any localization of a domain D is a flat overring of D . Thus, we can easily get the following result as a corollary of Corollary 3.7.

Corollary 3.10. *A quasi-stable domain D is locally quasi-stable (i.e., D_P is quasi-stable for each $P \in \text{Spec}(D)$).*

For the inverse implication, that is whether a locally quasi-stable domain is quasi-stable, we give partial results.

We recall that a domain D is h -local if each nonzero ideal I of D is contained in at most finitely many maximal ideals of D and each nonzero prime ideal of D is contained in a unique maximal ideal of D . Examples of h -local domains are one-dimensional Noetherian domains or domains in which each nonzero ideal is divisorial [11,29].

We show that if a domain D is locally quasi-stable and h -local, then D is quasi-stable. Note that this does not allow us to reduce the problem of flat-stability to the local case, because quasi-stable domains are not necessarily h -local (a Prüfer domain is quasi-stable but it may not be h -local).

Lemma 3.11. *Let D be an integral domain and I a nonzero ideal of D . Assume that $(I : I)D_M = (ID_M : ID_M)$, for all $M \in \text{Max}(D)$. If ID_M is quasi-stable (as an ideal of D_M) for all $M \in \text{Max}(D)$, then I is quasi-stable.*

Proof. We need to show that $I(A \cap B) = IA \cap IB$, for each $A, B \in \mathbf{F}(I : I)$. This is equivalent to showing that $I_M(A_M \cap B_M) = I_M A_M \cap I_M B_M$, for each $M \in \text{Max}(D)$. But $A_M, B_M \in \mathfrak{F}((I : I)_M)$ and since, by hypothesis $(I : I)D_M = (ID_M : ID_M)$, A_M, B_M are $(ID_M : ID_M)$ -modules. So $I_M(A_M \cap B_M) = I_M A_M \cap I_M B_M$ because ID_M is flat over $(ID_M : ID_M)$. \square

Note that the equality $(I : I)D_M = (ID_M : ID_M)$ is always satisfied when I is finitely generated, by the flatness of D_M over D . But this case is not interesting since quasi-stable finitely generated ideals are stable (and have already been widely studied especially in the finitely generated case, cf. [19,33]).

In general, as the following example shows, it may happen that $(I : I)D_M \neq (ID_M : ID_M)$ even in quasi-stable domains.

Example 3.12. Consider the domain $\text{Int}(\mathbb{Z}) := \{f(X) \in \mathbb{Q}[X] \mid f(\mathbb{Z}) \subseteq \mathbb{Z}\}$. It is well known that $\text{Int}(\mathbb{Z})$ is completely integrally closed, being \mathbb{Z} completely integrally closed [10, Proposition VI.2.1]. Thus, $(I : I) = \text{Int}(\mathbb{Z})$, for each nonzero ideal I of $\text{Int}(\mathbb{Z})$. It is also well known that $\text{Int}(\mathbb{Z})$ is a two-dimensional Prüfer domain [10], whence there exists a maximal ideal M such that $\text{Int}(\mathbb{Z})_M$ is a two-dimensional valuation domain. It follows that $\text{Int}(\mathbb{Z})_M$ is not completely integrally closed and so there exists a nonzero ideal I of $\text{Int}(\mathbb{Z})$ such that $(I_M : I_M) \neq \text{Int}(\mathbb{Z})_M$. But, $\text{Int}(\mathbb{Z}) = (I : I)$, so we have that $(I_M : I_M) \neq (I : I)_M$.

Olberding [29, Lemma 3.8] has shown that if D is h -local, then the equality $(I : I)D_M = (ID_M : ID_M)$ holds, for each $I \in \mathbf{F}(D)$ and $M \in \text{Max}(D)$. Then, for h -local domains, the quasi-stable property can be locally verified.

Corollary 3.13. *Let D be an h -local domain. Then D is quasi-stable if and only if D_M is quasi-stable for each $M \in \text{Max}(D)$.*

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