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Strong Mori and Noetherian properties of integer-valued polynomial rings

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Abstract

Let D be a domain with quotient field K and let $\mathrm{Int}(D)$ be the ring of integer-valued polynomials $\{f \in K[X] \mid f(D) \subseteq D\}$. We give conditions on D so that the ring $\mathrm{Int}(D)$ is a Strong Mori domain. In particular, we give a complete characterization in the case that the conductor $(D:D')$ is nonzero, where D' is the integral closure of D. We also show that when D is quasilocal with $\mathrm{Int}(D) \neq D[X]$ or D is Noetherian, $\mathrm{Int}(D)$ is a Strong Mori domain if and only if $\mathrm{Int}(D)$ is Noetherian.

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1. Introduction

Throughout D is an integral domain with quotient field K. To avoid trivial cases we assume that D is not a field.

An ideal J of D is called a *Glaz-Vasconcelos ideal* (in short, a GV-ideal) if J is finitely generated and $(D:J)=D$. The set of Glaz-Vasconcelos ideals of D is denoted by $\mathrm{GV}(D)$. Given a nonzero fractional ideal I of D, the *w-closure* of I is the ideal

$$I_w = \{x \in K \mid xJ \subseteq I \text{ for some } J \in \mathrm{GV}(D)\}.$$

Let $\mathscr{F}(D)$ be the set of nonzero fractional ideals of D. The *w-operation* $w:\mathscr{F}(D) \to \mathscr{F}(D)$, defined by $I \mapsto I_w$, is a $*$-operation of finite character, that is, $I_w = \bigcup\{J_w \mid J$

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is a finitely generated subideal of $I\}$. A fractional ideal $I \in \mathscr{F}(D)$ is called a *w-ideal* if $I = I_w$, and a prime w-ideal is called a *w-prime ideal*. An integral ideal which is maximal among the proper w-ideals of D is called *w-maximal* and it is a prime ideal. A fractional ideal $I \in \mathscr{F}(D)$ is said to be of *w-finite type* if there exists a finitely generated subideal J of I such that $J_w = I_w$. We will denote by $\max(D)$ the set of maximal ideals of D and by w-$\max(D)$ the set of w-maximal ideals of D. Then it is known that for any domain D, we have $D = \bigcap_{p \in w\text{-}\max(D)} D_p$. For more details about the w-operation the readers are referred to [6,7].

An integral domain D is called a *Strong Mori domain* (in short, an SM domain) if it satisfies the ascending chain condition (a.c.c.) on integral w-ideals. Thus, the class of Strong Mori domains includes Noetherian domains. Recall that an integral domain D is a Mori domain if it satisfies the a.c.c. on integral divisorial ideals. Since each divisorial ideal is a w-ideal, Strong Mori domains form a subclass of Mori domains.

It is interesting to notice that SM domains "replicate" in a weaker form many important properties of Noetherian domains. For instance, if D is an SM domain, $D[X]$ is also an SM domain; D is an SM domain if and only if every w-prime ideal is of w-finite type; if D is an SM domain, then the primary decomposition property for w-ideals holds, etc. (cf. [6,7]). We remark that these properties do not hold in general for Mori domains.

The *integer-valued polynomial ring* over a domain D is defined as follows:

$$\operatorname{Int}(D) = \{f \in K[X] \mid f(D) \subseteq D\}.$$

It is clear that $D[X] \subseteq \operatorname{Int}(D) \subseteq K[X]$. Establishing when $\operatorname{Int}(D)$ is Noetherian is an open question. In fact, the Noetherian property does not transfer from D to $\operatorname{Int}(D)$. The simplest example is $\operatorname{Int}(\mathbb{Z})$ [3, Proposition V.2.7]. In [11], the authors deeply study the Noetherian property for $\operatorname{Int}(D)$ by giving some necessary and some sufficient conditions. Among other results, they prove that if $\operatorname{Int}(D)$ is Noetherian then $\operatorname{Int}(D) \subseteq D'[X]$, where D' is the integral closure of D. They also show that if D is one-dimensional or integrally closed then $\operatorname{Int}(D)$ is Noetherian if and only if D is Noetherian and $\operatorname{Int}(D) = D[X]$. But, in general, the question is still unsolved.

In this paper we give conditions for $\operatorname{Int}(D)$ to be an SM domain. It turns out that some conditions already known for the Noetherian problem, such as $\operatorname{Int}(D) \subseteq D'[X]$, are still necessary for the SM property (Proposition 3.1). Moreover, we find other necessary conditions (Proposition 1.4), which would allow us to make progress on the Noetherian question.

In some cases, we get equivalence between the Noetherian and the SM properties for $\operatorname{Int}(D)$. For instance, when D is quasilocal and $\operatorname{Int}(D) \neq D[X]$ (Theorem 2.4) or when D is Noetherian (Theorem 3.3) it occurs that $\operatorname{Int}(D)$ is an SM domain if and only if $\operatorname{Int}(D)$ is Noetherian. More generally, the question concerning the SM property of $\operatorname{Int}(D)$ is somehow related to the problem concerning the Noetherian property of $\operatorname{Int}(D_2)$, where D_2 is a suitable overring of D (Theorem 3.4).

Lastly, in the case that $(D : D') \neq (0)$, we give a complete characterization of the domains D such that $\operatorname{Int}(D)$ is an SM domain (Theorem 3.2).

Some necessary conditions

We recall some useful results on SM domains, which will be used frequently.

Theorem 1.1 (Fanggui and McCasland ([6, Theorem 4.3] and [7, Theorem 1.9])). *D is a Strong Mori domain if and only if one of the following equivalent conditions holds*:

(1) *every w-ideal is of w-finite type*;
(2) *every w-prime ideal is of w-finite type*;
(3) *D_p is Noetherian for every $p \in w$-$\max(D)$ and $D = \bigcap_{p \in w\text{-}\max(D)} D_p$ has finite character*.

Theorem 1.2 (Park [16, Theorem 2.2 and Propositioon 3.2]). *Let $\mathscr{S}, \mathscr{S}_i (i \in I)$ be generalized multiplicative systems of D.*

(1) *If D is a Strong Mori domain, then the generalized quotient ring $D_{\mathscr{S}}$ of D is also a Strong Mori domain.*
(2) *If $D = \bigcap_{i \in I} D_{\mathscr{S}_i}$ has finite character and each $D_{\mathscr{S}_i}$ is a Strong Mori domain, then D is also a Strong Mori domain.*

We recall that if S is a multiplicative subset of any domain D, then $\operatorname{Int}(D)_s \subseteq \operatorname{Int}(D_S)$ [3, Proposition I.2.2], and in particular, if D is a Mori domain, then $\operatorname{Int}(D)_S = \operatorname{Int}(D_S)$ [4, Proposition 2.1]. Henceforth, we will freely use the fact that if p is a prime ideal of a Mori domain D, then $\operatorname{Int}(D)_{(D \setminus p)} := \operatorname{Int}(D)_p = \operatorname{Int}(D_p)$, and if p has infinite residue field, then $\operatorname{Int}(D_p) = D_p[X]$ [3, Remarks I.3.5].

It is known that if $\operatorname{Int}(D)$ is a Mori domain (resp., Noetherian), then D is a Mori domain (resp., Noetherian) [4, Proposition 2.6] (resp., [3, p. 131]). In the following proposition we will see that the analogue of these results holds for the SM property too.

Notation. If J is a subset of $K(X)$ and $a \in K$, we set $J(a) = \{\varphi(a) \mid \varphi \in J\}$. It is easy to see that $\operatorname{Int}(D)(a) = D$ for any $a \in D$.

Proposition 1.3. *If* $\operatorname{Int}(D)$ *is a Strong Mori domain, then D is a Strong Mori domain.*

Proof. It is enough to verify that every w-prime ideal of D is of w-finite type. Take a w-prime ideal p of D and consider the ideal $p\operatorname{Int}(D)$. Since $\operatorname{Int}(D)$ is an SM domain, $p\operatorname{Int}(D)$ is of w-finite type. So there exists a finitely generated subideal I of p such that $(p\operatorname{Int}(D))_w = (I\operatorname{Int}(D))_w$.

We show that $p = I_w$. Obviously, $I_w \subseteq p$. Conversely, let $x \in p$. Then $x \in (p\operatorname{Int}(D))_w = (I\operatorname{Int}(D))_w$, whence $xJ \subseteq I \operatorname{Int}(D)$ for some $J \in \operatorname{GV}(\operatorname{Int}(D))$. For each $a \in D$ we have that $J(a) \subseteq D$ and $xJ(a) \subseteq I$. Let J' be the ideal of D generated by $\bigcup_{a \in D} J(a)$. Since D is a Mori domain [4, Proposition 2.6], there exists a finite subset $\{c_1, \ldots, c_n\}$ of $\bigcup_{a \in D} J(a)$ such that $J'_v = (c_1, \ldots, c_n)_v$. Since $xJ' \subseteq I, x(c_1, \ldots, c_n) \subseteq I$.

We claim that $(c_1,\dots,c_n)\in \mathrm{GV}(D)$. Let $y\in(c_1,\dots,c_n)^{-1}=J'^{-1}$. Then $yJ'\subseteq D$, and so $yf(a)\in D$ for all $f\in J$ and all $a\in D$. Therefore, $yf\in \mathrm{Int}(D)$ for all $f\in J$, whence $yJ\subseteq \mathrm{Int}(D)$. Since $J\in \mathrm{GV}(\mathrm{Int}(D))$, $y\in J^{-1}=\mathrm{Int}(D)$. Thus, we have $y\in \mathrm{Int}(D)\cap K=D$.

Therefore, $x\in I_w$, and hence we have $p\subseteq I_w$. □

Let I be an ideal of D. We denote by $\mathrm{Int}(D,I)$ the set $\{f\in K[X]\mid f(D)\subseteq I\}$. Clearly, $\mathrm{Int}(D,I)$ is an ideal of $\mathrm{Int}(D)$.

Proposition 1.4. *If p is a prime ideal of any domain D such that $\mathrm{Int}(D_p)\neq D_p[X]$, then p is a w-prime ideal with finite residue field. If, in addition, $\mathrm{Int}(D)$ is a Strong Mori domain, then there exist only finitely many prime ideals p of D such that $\mathrm{Int}(D_p)\neq D_p[X]$ and they all have height >1.*

Proof. Let p be a prime ideal of D such that $\mathrm{Int}(D_p)\neq D_p[X]$. By Cahen and Chabert [3, Corollary I.3.7] p has finite residue field and it is maximal.

We claim that p is a w-prime ideal. Suppose not. Then pD_p is not a w-prime ideal of D_p (otherwise $p=pD_p\cap D$ would be a w-ideal by Fanggui and McCasland [7, Lemma 3.1] and Park [16, Lemma 2.1]). We can write $D_p=\bigcap_{Q\in w\text{-}\max(D_p)}(D_p)_Q$. It follows that $D_p=\bigcap_{q\subsetneq p}D_q$. Every prime ideal q properly contained in p is non-maximal, whence it has infinite residue field and $\mathrm{Int}(D_q)=D_q[X]$. Thus, we have $\mathrm{Int}(D_p)=\bigcap_{q\subsetneq p}\mathrm{Int}(D_q)=\bigcap_{q\subsetneq p}D_q[X]=D_p[X]$ (where the first equality holds by Cahen and Chabert [2, Corollaires 3]), which is against our assumption.

Now, assume that $\mathrm{Int}(D)$ is an SM domain. Then $\mathrm{Int}(D,p)=\{f\in K[X]\mid f(D)\subseteq p\}$ is an ideal of $\mathrm{Int}(D)$. By Cahen and Chabert [3, Lemma V.1.9], every prime ideal of $\mathrm{Int}(D)$ containing $\mathrm{Int}(D,p)$ is maximal. Since p is w-maximal, it is t-maximal [15, Lemma 2.1]. Moreover, since D is a Mori domain (Proposition 1.3), p is also divisorial. Hence, $\mathrm{Int}(D,p)$ is divisorial [4, Lemma 4.1]. Let $\mathscr{P}_0=\{f\in \mathrm{Int}(D)\mid f(0)\in p\}$. Then $\mathrm{Int}(D,p)\subset\mathscr{P}_0$ and $\mathscr{P}_0$ is minimal over $\mathrm{Int}(D,p)$. Therefore, $\mathscr{P}_0$ is t-maximal, and hence it is w-maximal.

Note that $X\in\mathscr{P}_0$ for any prime ideal p. Since $\mathrm{Int}(D)$ is an SM domain, the representation $\mathrm{Int}(D)=\bigcap_{Q\in w\text{-}\max(\mathrm{Int}(D))}\mathrm{Int}(D)_Q$ has finite character, and hence we could have only finitely many ideals of the type $\mathscr{P}_0$. Therefore, there exist only finitely many prime ideals p of D with $\mathrm{Int}(D_p)\neq D_p[X]$.

Lastly, if $\mathrm{ht}(p)=1$, then D_p is a one-dimensional Noetherian domain and $\mathrm{Int}(D_p)=\mathrm{Int}(D)_p$ is an SM domain by Theorem 1.2, and hence $\mathrm{Int}(D_p)=D_p[X]$ by Cahen et al. [4, Theorem 3.1]. Therefore, $\mathrm{ht}(p)>1$. □

We recall that the *w-dimension* of D (which is denoted by w-dim(D)) is defined as the supremum of lengths of chains of w-prime ideals and the zero ideal (0). Any nonzero prime ideal that is contained in a w-prime ideal is a w-prime ideal [7, Proposition 1.1], so we have $w\text{-}\dim(D)=\sup\{\mathrm{ht}(p)\mid p\in w\text{-}\max(D)\}$. Since one-dimensional Noetherian domains and Krull domains are SM domains of w-dimension 1 [7, Theorem 2.8], we have the following generalization of [11, Corollaires 2.4 and 2.5] and [4, Corollary 2.7 (2)$\Leftrightarrow$(3)]:

Corollary 1.5. *Let D be a domain of w-dimension* 1. *Then* $\mathrm{Int}(D)$ *is a Strong Mori domain if and only if D is a Strong Mori domain and* $\mathrm{Int}(D) = D[X]$.

Proof. Assume that $\mathrm{Int}(D)$ is an SM domain. By Proposition 1.3, D is an SM domain. Since w-dim$(D)=1$, each w-prime ideal p of D has height 1. Thus, by Proposition 1.4, $\mathrm{Int}(D_p) = D_p[X]$, and hence

$$\mathrm{Int}(D) = \bigcap_{p\in w\text{-}\max(D)} \mathrm{Int}(D_p) = \bigcap_{p\in w\text{-}\max(D)} D_p[X] = D[X]$$

(where the first equality holds by Cahen and Chabert [2, Corollaires 3]. Conversely, if $\mathrm{Int}(D) = D[X]$ and D is an SM domain, then $D[X]$ is an SM domain by Fanggui and McCasland [7, Theorem 1.13]. $\square$

Consider the following decomposition of the set $\mathrm{Spec}(D)$ of prime ideals of D:

$$\mathscr{P}_1 := \{p \in \mathrm{Spec}(D) \mid \mathrm{Int}(D_p) = D_p[X]\},$$

$$\mathscr{P}_2 := \{p \in \mathrm{Spec}(D) \mid \mathrm{Int}(D_p) \neq D_p[X]\}.$$

Set $D_1 = \bigcap_{p\in\mathscr{P}_1} D_p$ and $D_2 = \bigcap_{p\in\mathscr{P}_2} D_p$ (where $D_i = K$ in case $\mathscr{P}_i = \emptyset$, $i = 1,2$). Then, $D = D_1 \cap D_2$ and from [2, Corollaires 3] and [5, Lemma 4.1], we have

$$\mathrm{Int}(D_1) = \bigcap_{p\in\mathscr{P}_1} \mathrm{Int}(D_p) = \bigcap_{p\in\mathscr{P}_1} D_p[X] = D_1[X],$$

$$\mathrm{Int}(D_2) = \bigcap_{p\in\mathscr{P}_2} \mathrm{Int}(D_p),$$

and

$$\mathrm{Int}(D) = \mathrm{Int}(D_1) \cap \mathrm{Int}(D_2) = D_1[X] \cap \mathrm{Int}(D_2).$$

We here observe that $\mathrm{Int}(D_i) = \bigcap_{p\in\mathscr{P}_i} \mathrm{Int}(D_p) = \bigcap_{p\in\mathscr{P}_i} \mathrm{Int}(D)_p$ [17, p. 3], $i = 1,2$, is a generalized quotient ring of $\mathrm{Int}(D)$.

Lemma 1.6. *With the above notation, if* $\mathrm{Int}(D)$ *is a Strong Mori domain, then D_2 is Noetherian.*

Proof. By definition and Proposition 1.4, $D_2 = D_{p_1} \cap \cdots \cap D_{p_m}$, where $p_1, \ldots, p_m$ are the prime ideals of D such that $\mathrm{Int}(D_{p_i}) \neq D_{p_i}[X]$. By Kaplansky [13, Theorem 105], the maximal ideals of D_2 are exactly $p_1D_2, \ldots, p_mD_2$. Each p_i is a w-prime ideal (Proposition 1.4), whence D_{p_i} is Noetherian. Finally, $D_{p_i} = (D_2)_{p_iD_2}$ and so D_2 is Noetherian by Kaplansky [13, Exercise 10, p. 73]. $\square$

Proposition 1.7. *Let D be a Strong Mori domain. With the above notation,* $\mathrm{Int}(D)$ *is a Strong Mori domain if and only if* $\mathrm{Int}(D_2)$ *is a Strong Mori domain.*

Proof. We have already observed that $\mathrm{Int}(D_i)$ is a generalized quotient ring of $\mathrm{Int}(D), i = 1, 2$. So, if $\mathrm{Int}(D)$ is an SM domain, then $\mathrm{Int}(D_2)$ is an SM domain by Theorem 1.2.

Conversely, assume that $\mathrm{Int}(D_2)$ is an SM domain. Since D_1 is a generalized quotient ring of the SM domain D, D_1 is an SM domain, and so $D_1[X]$ is also an SM domain [7, Theorem 1.13]. Therefore, by Theorem 1.2, $\mathrm{Int}(D) = D_1[X] \cap \mathrm{Int}(D_2)$ is an SM domain. □

The problem is now reduced to studying $\mathrm{Int}(D_2)$, which we call the *non polynomial part* of $\mathrm{Int}(D)$. Applying the same argument as in Propositions 1.7 and 1.4, we can say that $\mathrm{Int}(D_2)$ is an SM domain if and only if $\mathrm{Int}(D_p)$ is an SM domain for each $p \in \mathscr{P}_2$ and $\mathscr{P}_2$ is finite. Therefore, we can focus our study on the local case.

2. Local case

Let (D, p) be a quasilocal domain such that $\mathrm{Int}(D) \neq D[X]$. In order that $\mathrm{Int}(D)$ be an SM domain, we have to assume that D is Noetherian and $\mathrm{ht}(p) > 1$ (Lemma 1.6 and Proposition 1.4). Hence, throughout this section, we will assume that (D, p) is a Noetherian quasilocal domain (in short, a local domain) such that $\dim(D) > 1$ and $\mathrm{Int}(D) \neq D[X]$, unless otherwise specified. By Proposition 1.4, we have that p is a w-prime ideal with finite residue field. Moreover, p is divisorial because p is a w-maximal ideal of the Mori domain D [15, Lemma 2.1].

Let $a \in D$; we consider the ideals $\mathscr{P}_a = \{f \in \mathrm{Int}(D) \mid f(a) \in p\}$. From [3, Remarks V.3.8 (iii)], it follows that $\{\mathscr{P}_a \mid a \in D\} = \{\mathscr{P}_{a_1}, \ldots, \mathscr{P}_{a_n}\}$, where $\{a_1, \ldots, a_n\}$ is a set of representatives of D modulo p. We also observe that

$$\bigcap_{j=1}^{n} \mathscr{P}_{a_j} = \bigcap_{a \in D} \mathscr{P}_a = \mathrm{Int}(D, p).$$

Lemma 2.1. *If I is an integral ideal of* $\mathrm{Int}(D)$ *such that $I \nsubseteq \mathscr{P}_{a_i}$ for each $i = 1, \ldots, n$, and $I \cap D \neq (0)$, then $I_v = \mathrm{Int}(D)$.*

Proof. Since $I^{-1} \subseteq \bigcap_{P \in \mathrm{Spec}(\mathrm{Int}(D)),\, I \nsubseteq P} \mathrm{Int}(D)_P$, we have that $I^{-1} \subseteq K[X] \cap (\bigcap_{a \in D} \mathrm{Int}(D)_{\mathscr{P}_a})$. Now,

$$\begin{aligned}\mathrm{Int}(D)_{\mathscr{P}_a} &= \left\{ \frac{f}{g} \mid f, g \in \mathrm{Int}(D), g \notin \mathscr{P}_a \right\} \\ &= \left\{ \frac{f}{g} \mid f, g \in \mathrm{Int}(D), g(a) \notin p \right\} \\ &\subseteq \{\varphi \in K(X) \mid \varphi(a) \in D\}.\end{aligned}$$

Hence,

$$\bigcap_{a\in D} \mathrm{Int}(D)_{\mathscr{P}_a} \subseteq \{\varphi \in K(X) \mid \varphi(D) \subseteq D\}$$

and

$$K[X] \cap \left(\bigcap_{a\in D} \mathrm{Int}(D)_{\mathscr{P}_a} \right) = \mathrm{Int}(D).$$

Therefore, $I^{-1} = \mathrm{Int}(D)$, and consequently, $I_v = \mathrm{Int}(D)$. □

Proposition 2.2. *Let (D, p) be a local domain as above. A nonzero prime ideal Q of $\mathrm{Int}(D)$ is a w-prime ideal if and only if Q is an upper to zero or $Q \subseteq \mathscr{P}_{a_i}$ for some $i = 1, \dots, n$.*

Proof. If Q is an upper to zero, then it is a prime ideal of height 1 and hence a t-ideal [12, Corollaires 3, p. 31]. Since each t-ideal is a w-ideal (which follows easily from the definition of the w-closure), Q is a w-ideal.

Let us suppose that $Q \cap D \neq (0)$. We start by proving that each ideal of the type $\mathscr{P}_{a_i}$, $i = 1, \dots, n$, is divisorial, and hence a w-prime ideal. Let $i (1 \leqslant i \leqslant n)$ be fixed. Choose a polynomial $f \in (\bigcap_{j\neq i, 1\leqslant j\leqslant n} \mathscr{P}_{a_j}) \backslash \mathscr{P}_{a_i}$. Since $\bigcap_{j=1}^{n} \mathscr{P}_{a_j} = \mathrm{Int}(D, p)$, we have that $\mathscr{P}_{a_i} = (\mathrm{Int}(D, p) : f)$, which is a divisorial ideal.

Now, if $Q \subseteq \mathscr{P}_{a_i}$ for some $i = 1, \dots, n$, then Q is a w-prime ideal by Fanggui and McCasland [7, Proposition 1.1].

If $Q \cap D \neq (0)$ and $Q \nsubseteq \mathscr{P}_{a_i}$ for any $i = 1, \dots, n$, then choose $f \in Q \backslash (\bigcup_{i=1,\dots,n} \mathscr{P}_{a_i})$ and a nonzero constant $c \in Q$. Let $I = (f, c)\,\mathrm{Int}(D)$, then it follows from Lemma 2.1 that $I_v = \mathrm{Int}(D)$, whence $I \in \mathrm{GV}(\mathrm{Int}(D))$ and $I \subseteq Q$. Thus Q is not a w-prime ideal. □

We have the following characterization of the w-maximal ideals of $\mathrm{Int}(D)$.

Corollary 2.3. *Let (D, p) be a local domain as above. The w-maximal ideals of* $\mathrm{Int}(D)$ *are the uppers to zero which are not contained in any ideals of the type $\mathscr{P}_a$, and the ideals $\mathscr{P}_a$ themselves, $a \in D$.*

The *global transform* of D is defined to be the set

$$D^g = \{x \in K \mid M_1 \cdots M_k x \subseteq D,\ M_i \in \max(D)\}$$

and it is an overring of D. It is well known that for D Noetherian, $D^g = \bigcap_{p\in \mathrm{Spec}(D)\backslash \max(D)} D_p$ and any ring T between D and D^g is Noetherian (see [14]).

If D is a nonNoetherian SM domain, then $D[X]$ is still a nonNoetherian SM domain. The following theorem shows that when D is quasilocal and $\mathrm{Int}(D) \neq D[X]$, the SM and the Noetherian properties are equivalent for $\mathrm{Int}(D)$.

Theorem 2.4. *Let (D, p) be a quasilocal domain such that* $\mathrm{Int}(D) \neq D[X]$. *Then* $\mathrm{Int}(D)$ *is a Strong Mori domain if and only if it is Noetherian.*

Proof. Obviously, if $\mathrm{Int}(D)$ is Noetherian, then it is an SM domain.

Now assume that $\mathrm{Int}(D)$ is an SM domain. By Lemma 1.6, D is Noetherian. We set $B=\bigcap_{q\in\mathrm{Spec}(D),q\neq p} D_q$. By construction, $B=D^g$, whence it is Noetherian. Since $\mathrm{Int}(D)$ is an SM domain and the ideals $\mathscr{P}_{a_i}$ are w-maximal, it follows that $\mathrm{Int}(D)_{\mathscr{P}_{a_i}}$ is Noetherian for each $i=1,\ldots,n$. Thus, $A:=\bigcap_{i=1}^{n}\mathrm{Int}(D)_{\mathscr{P}_{a_i}}$ is Noetherian (by the same argument as in Lemma 1.6). Moreover, using the same argument as in Lemma 2.1, we have that $\mathrm{Int}(D)=K[X]\cap A$. For each prime ideal $q(\neq p)$ of D, $\mathrm{Int}(D)\subseteq D_q[X]\subseteq K[X]$, so $\mathrm{Int}(D)=(\bigcap_{q\in\mathrm{Spec}(D),q\neq p} D_q[X])\cap A=B[X]\cap A$.

We claim that $B[X]\subseteq\bigcap_{\mathfrak{q}\in\max(\mathrm{Int}(D)),\mathfrak{q}\neq\mathscr{P}_{a_i}}\mathrm{Int}(D)_{\mathfrak{q}}$. Let $\mathfrak{q}$ be a maximal ideal of $\mathrm{Int}(D)$ which is not of the form $\mathscr{P}_{a_i}$, $i=1,\ldots,n$. Then

$$\begin{aligned}\mathrm{Int}(D)_{\mathfrak{q}} &= \bigcap_{Q'\in w\text{-}\max(\mathrm{Int}(D)_{\mathfrak{q}})}(\mathrm{Int}(D)_{\mathfrak{q}})_{Q'}\\ &= \bigcap_{Q\in\mathrm{Spec}(\mathrm{Int}(D)),Q_{\mathfrak{q}}\in w\text{-}\max(\mathrm{Int}(D)_{\mathfrak{q}})}\mathrm{Int}(D)_Q.\end{aligned}$$

Now, $Q_{\mathfrak{q}}$ is a divisorial ideal of $\mathrm{Int}(D)_{\mathfrak{q}}$ (since it is a w-maximal ideal of the SM domain $\mathrm{Int}(D)_{\mathfrak{q}}$). Suppose that $Q\cap D=p$, then

$$Q_{\mathfrak{q}}\supseteq(p\ \mathrm{Int}(D)_{\mathfrak{q}})_v=(p\ \mathrm{Int}(D))_v\mathrm{Int}(D)_{\mathfrak{q}},$$

because $\mathrm{Int}(D)$ is a Mori domain. Since

$$(p\ \mathrm{Int}(D))_v\,\mathrm{Int}(D)_{\mathfrak{q}}=\mathrm{Int}(D,p)\,\mathrm{Int}(D)_{\mathfrak{q}}=\left(\bigcap_{i=1}^{n}\mathscr{P}_{a_i}\right)\mathrm{Int}(D)_{\mathfrak{q}}=\mathrm{Int}(D)_{\mathfrak{q}},$$

we get that $Q_{\mathfrak{q}}\supseteq\mathrm{Int}(D)_{\mathfrak{q}}$, which is a contradiction. Thus, $Q\cap D=q\subsetneq p$ and $\mathrm{Int}(D)_Q\supseteq D_{\mathfrak{q}}[X]\supseteq B[X]$, hence $\mathrm{Int}(D)_{\mathfrak{q}}\supseteq B[X]$.

Let P be a prime ideal of $\mathrm{Int}(D)$. Since $B[X]$ and A are Noetherian rings, there exists a finitely generated subideal I of P such that $PB[X]=IB[X]$ and $PA=IA$. Now,

$$\begin{aligned}P &\subseteq PB[X]\cap PA\\ &= IB[X]\cap IA\\ &\subseteq\left(\bigcap_{\mathfrak{q}\in\max(\mathrm{Int}(D)),\,\mathfrak{q}\neq\mathscr{P}_{a_i},i=1,\ldots,n} I\,\mathrm{Int}(D)_{\mathfrak{q}}\right)\cap\left(\bigcap_{i=1}^{n} I\,\mathrm{Int}(D)_{\mathscr{P}_{a_i}}\right)\\ &= \bigcap_{\mathfrak{q}\in\max(\mathrm{Int}(D))} I\,\mathrm{Int}(D)_{\mathfrak{q}}\\ &= I.\end{aligned}$$

Therefore, $P=I$ is finitely generated. It follows that $\mathrm{Int}(D)$ is Noetherian. □

As an immediate corollary, we obtain the following result:

Corollary 2.5. *Let* (D, p) *be a local domain with finite residue field and* $\dim(D) > 1$. *Then the following conditions are equivalent*:

(1) $\mathrm{Int}(D)$ *is Noetherian*,
(2) $\mathrm{Int}(D)$ *is a Strong Mori domain*,
(3) $\mathrm{Int}(D)_{\mathscr{P}_{a_i}}$ *is Noetherian for each* $i = 1, \ldots, n$,
(4) $\mathrm{Int}(D)_{\mathfrak{U}} = \bigcap_{i=1}^{n} \mathrm{Int}(D)_{\mathscr{P}_{a_i}}$ *is Noetherian*, *where* $\mathfrak{U} = \{f \in \mathrm{Int}(D) | f(D) \subseteq D \setminus p\}$.

3. Global case

We denote by D' the integral closure of D. It is well known that if $\mathrm{Int}(D)$ is Noetherian, then $\mathrm{Int}(D) \subseteq D'[X]$ ([11, Theorem 2.3] and [3, Corollary IV.4.10]). We show that the same necessary condition holds assuming that $\mathrm{Int}(D)$ is an SM domain.

Proposition 3.1. *Let D be a domain. If* $\mathrm{Int}(D)$ *is a Strong Mori domain*, *then* $\mathrm{Int}(D) \subseteq D'[X]$.

Proof. If $\mathrm{Int}(D)$ is an SM domain, then $\mathscr{P}_2 = \{p \in \mathrm{Spec}(D) \mid \mathrm{Int}(D_p) \neq D_p[X]\}$ is finite, say $\mathscr{P}_2 = \{p_1, \ldots, p_m\}$. Then each ring D_{p_i}, $i = 1, \ldots, m$, is a local domain of the type studied in Section 2. Moreover, if $\mathrm{Int}(D)$ is an SM domain, then $\mathrm{Int}(D)_{p_i} = \mathrm{Int}(D_{p_i})$ is an SM domain. Hence, from Theorem 2.4, we have that $\mathrm{Int}(D_{p_i})$ is Noetherian and so $\mathrm{Int}(D_{p_i}) \subseteq D'_{p_i}[X]$ ([11, Theorem 2.3] and [3, Corollary IV.4.10]).

Thus,

$$
\begin{aligned}
\mathrm{Int}(D) &= D_1[X] \cap \left(\bigcap_{i=1}^{m} \mathrm{Int}(D_{p_i}) \right) \\
&\subseteq \left(\bigcap_{p \in \max(D), p \neq p_i} D[X]_p \right) \cap \left(\bigcap_{i=1}^{m} D'_{p_i}[X] \right) \\
&\subseteq \left(\bigcap_{p \in \max(D), p \neq p_i} D'[X]_p \right) \cap \left(\bigcap_{i=1}^{m} D'[X]_{p_i} \right) \\
&= \bigcap_{p \in \max(D)} D'[X]_p \\
&= D'[X]. \qquad \square
\end{aligned}
$$

We recall that given a domain D, the *conductor* of D' in D is the ideal $I = (D : D') = \{x \in K | xD' \subseteq D\}$.

Theorem 3.2. *Let D be a domain such that $(D:D') \neq (0)$. Then $\mathrm{Int}(D)$ is a Strong Mori domain if and only if the following three conditions hold*:

(1) *D is a Strong Mori domain*;
(2) $\mathrm{Int}(D) \subseteq D'[X]$;
(3) *there exist only finitely many prime ideals p of D such that $\mathrm{Int}(D_p) \neq D_p[X]$.*

Proof. If $\mathrm{Int}(D)$ is an SM domain, then the three necessary conditions follow from Propositions 1.3, 3.1 and 1.4.

Conversely, assume that (1)–(3) hold. Following the notation of Section 1, we write $\mathrm{Int}(D) = D_1[X] \cap \mathrm{Int}(D_2)$. Then, since D is an SM domain, D_1 is an SM domain, and so $D_1[X]$ is also an SM domain. Now let $\mathscr{P}_2 = \{p \in \mathrm{Spec}(D) \mid \mathrm{Int}(D_p) \neq D_p[X]\} = \{p_1, \dots, p_m\}$, then

$$D_2[X] \subseteq \mathrm{Int}(D_2) = \bigcap_{i=1}^{m} \mathrm{Int}(D_{p_i}) = \bigcap_{i=1}^{m} \mathrm{Int}(D)_{p_i} \subseteq \bigcap_{i=1}^{m} D'_{p_i}[X] = (D_2)'[X].$$

Since $(D:D') \neq (0)$, $(D_2:(D_2)') \neq (0)$. Following the same argument as in Lemma 1.6, we can show that D_2 is Noetherian, whence $(D_2)'$ is a finitely generated D_2-module. It follows that $\mathrm{Int}(D_2)$ is a finitely generated $D_2[X]$-module and so $\mathrm{Int}(D_2)$ is Noetherian, whence it is an SM domain. Thus, $\mathrm{Int}(D)$ is the intersection of two SM domains which are generalized quotient rings of $\mathrm{Int}(D)$. Therefore, $\mathrm{Int}(D)$ is an SM domain. □

Theorem 3.3. *Let D be a Noetherian domain. Then $\mathrm{Int}(D)$ is Noetherian if and only if $\mathrm{Int}(D)$ is a Strong Mori domain.*

Proof. If $\mathrm{Int}(D)$ is Noetherian, then it is an SM domain.

Now assume that $\mathrm{Int}(D)$ is an SM domain. By Proposition 1.4 there exist only finitely many prime ideals of D, $p_1, \dots, p_m$, such that $\mathrm{Int}(D_{p_i}) \neq D_{p_i}[X]$. We set $B = \bigcap_{\mathfrak{q} \in \mathrm{Spec}(D) \setminus \{p_1, \dots, p_m\}} D_{\mathfrak{q}}$. By construction, $D \subseteq B \subseteq D^g$, whence B is Noetherian. Moreover, each D_{p_i} is a local domain of the type studied in Section 2 and $\mathrm{Int}(D_{p_i}) = \mathrm{Int}(D)_{p_i}$ is an SM domain. Thus, by Theorem 2.4, $\mathrm{Int}(D_{p_i})$ is Noetherian. We can write

$$\mathrm{Int}(D) = B[X] \cap \left(\bigcap_{i=1}^{m} \mathrm{Int}(D_{p_i}) \right),$$

where both the rings $B[X]$ and $\bigcap_{i=1}^{m} \mathrm{Int}(D_{p_i})$ are Noetherian.

Let Q be a prime ideal of $\mathrm{Int}(D)$. Then, there exists a finitely generated subideal I of Q such that $QB[X] = IB[X]$ and $Q(\bigcap_{i=1}^{m} \mathrm{Int}(D_{p_i})) = I(\bigcap_{i=1}^{m} \mathrm{Int}(D_{p_i}))$. Thus,

$$\begin{aligned} Q &\subseteq QB[X] \cap Q \left(\bigcap_{i=1}^{m} \mathrm{Int}(D_{p_i}) \right) \\ &= IB[X] \cap I \left(\bigcap_{i=1}^{m} \mathrm{Int}(D_{p_i}) \right) \end{aligned}$$

$$\subseteq \left(\bigcap_{\mathfrak{q}\in \mathrm{Spec}(D), \mathfrak{q}\neq \mathfrak{p}_1,\dots,\mathfrak{p}_m} I D_{\mathfrak{q}}[X] \right) \cap \left(\bigcap_{i=1}^{m} I \,\mathrm{Int}(D_{\mathfrak{p}_i}) \right)$$

$$= \bigcap_{\mathfrak{q}\in \mathrm{Spec}(D)} I \,\mathrm{Int}(D)_{\mathfrak{q}}$$

$$= I.$$

Therefore, $Q = I$ is finitely generated, and $\mathrm{Int}(D)$ is Noetherian. □

Now we can strengthen Proposition 1.7 as follows:

Theorem 3.4. *Let D be a domain. Then* $\mathrm{Int}(D)$ *is a Strong Mori domain if and only if D is a Strong Mori domain and* $\mathrm{Int}(D_2)$ *is Noetherian.*

Proof. By Propositions 1.3 and 1.7, $\mathrm{Int}(D)$ is an SM domain if and only if D is an SM domain and $\mathrm{Int}(D_2)$ is an SM domain. Hence, the conclusion follows from Lemma 1.6 and Theorem 3.3. □

We end this paper with an example of a nontrivial integer-valued polynomial ring $\mathrm{Int}(D)$ (i.e. $\mathrm{Int}(D) \neq D[X]$) which is an SM domain but not Noetherian.

Example 3.5. Let T be a nonNoetherian, Krull domain with infinite residue fields at all the height-one prime ideals, and let T have a maximal ideal $\mathfrak{p}$ such that $T_{\mathfrak{p}}$ is Noetherian and $|T/\mathfrak{p}| < \infty$. Suppose, also, that there exists a finite field k strictly contained in $T/\mathfrak{p}$. (Such a domain T does exist. If p_1 is a prime number, then there exists a nonfinitely generated torsion-free abelian group G of rank two such that each rank one subgroup of G is cyclic and such that G/H is a p_1-group for some finitely generated subgroup H of G (see [9, Vol. II, pp. 125]). For a prime number p_2 distinct from p_1, let F be a finite field of characteristic p_2 such that $F \supsetneq \mathbb{Z}_{p_2}$. Now let T be the group ring $F[X;G]$ and let $\mathfrak{p}$ be the maximal ideal of T generated by $\{1 - X^g \mid g \in G\}$. By Brewer et al. [1, Theorem C] and Gilmer [10, Theorem 3], T and $\mathfrak{p}$ have the desired properties.)

Let D be defined by the following pullback diagram:

$$\begin{array}{ccc} D & \longrightarrow & k \\ \downarrow & & \downarrow \\ T & \longrightarrow & T/\mathfrak{p}. \end{array}$$

Then D is a nonNoetherian [8, Proposition 1.8], SM domain [15, Proposition 3.7] with $\mathrm{Int}(D) \neq D[X]$ [4, Proposition 2.4]. Since D is a homomorphic image of $\mathrm{Int}(D)$ and D is not Noetherian, it follows that $\mathrm{Int}(D)$ is not Noetherian.

We get this other pullback diagram:

$$\begin{array}{ccc}
\mathrm{Int}(D) & \longrightarrow & \mathrm{Int}(D)/\mathrm{Int}(D,p) \\
\Big\downarrow & & \Big\downarrow \\
\mathrm{Int}(D,T) & \longrightarrow & \mathrm{Int}(D,T)/\mathrm{Int}(D,p)
\end{array}$$

where

$$\mathrm{Int}(D,T)=\{f\in K[X]\mid f(D)\subseteq T\},\ \mathrm{Int}(D,p)=\{f\in K[X]\mid f(D)\subseteq p\}.$$

Let $X'(T)$ be the set of height-one prime ideals of T. If $q'\in X'(T)$ and $q=q'\cap D$, then since $p\nsubseteq q'$, we have that $T_{q'}=D_q$ [8, Theorem 1.4(c)].

Thus,

$$\begin{aligned}
\mathrm{Int}(D,T) &= \mathrm{Int}\left(D, \bigcap_{q'\in X'(T)} T_{q'}\right) = \bigcap_{q'\in X'(T)} \mathrm{Int}(D,T_{q'}) \\
&= \bigcap_{q=q'\cap D,\, q'\in X'(T)} \mathrm{Int}(D_q,T_{q'}) = \bigcap_{q'\in X'(T)} \mathrm{Int}(T_{q'}) \\
&= \bigcap_{q'\in X'(T)} T_{q'}[X] = T[X].
\end{aligned}$$

where $\mathrm{Int}(T_{q'})=T_{q'}[X]$, because q' has infinite residue field in T.

We claim that $\mathrm{Int}(D)$ is an SM domain.

Let Q be a prime ideal of $\mathrm{Int}(D)$ such that $\mathrm{Int}(D,p)\nsubseteq Q$. Then, there exists a prime ideal Q' of $T[X]$ with $\mathrm{Int}(D)_Q=T[X]_{Q'}$ [8, Theorem 1.4(c)]. Thus

$$\begin{aligned}
\mathrm{Int}(D)_1 &:= \bigcap_{Q\in \mathrm{Spec}(\mathrm{Int}(D)),\mathrm{Int}(D,p)\nsubseteq Q} \mathrm{Int}(D)_Q \\
&= \bigcap_{Q'\in \mathrm{Spec}(T[X]),\mathrm{Int}(D,p)\nsubseteq Q'} T[X]_{Q'},
\end{aligned}$$

which is a generalized quotient ring of the Krull domain $T[X]$, and hence $\mathrm{Int}(D)_1$ is a Krull domain and so an SM domain.

On the other hand, take a prime ideal Q of $\mathrm{Int}(D)$ such that $\mathrm{Int}(D,p)\subseteq Q$. Then $Q\cap D=p$. Thus, $\mathrm{Int}(D)=\mathrm{Int}(D)_1\cap \mathrm{Int}(D)_p=\mathrm{Int}(D)_1\cap \mathrm{Int}(D_p)$. Note that $\mathrm{Int}(D)_1$ is a generalized quotient ring of $\mathrm{Int}(D)$. So, in view of Theorem 1.2, it is sufficient to show that $\mathrm{Int}(D_p)$ is an SM domain to have that $\mathrm{Int}(D)$ is an SM domain.

We claim that $\mathrm{Int}(D_p)$ is Noetherian. Note that

$$D_p[X]\subseteq \mathrm{Int}(D_p)=\mathrm{Int}(D)_p\subseteq \mathrm{Int}(D,T)_p=T[X]_p=T_p[X].$$

Since T_p is a finite D_p-module by construction (because $k \hookrightarrow T/p$ is a finite extension), $T_p[X]$ is a finite $D_p[X]$-module, and hence a finite $\mathrm{Int}(D_p)$-module. Therefore, from the assumption that T_p is Noetherian, it follows that $\mathrm{Int}(D_p)$ is Noetherian.

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References

[1] J.W. Brewer, D.L. Costa, E.L. Lady, Prime ideals and localization in commutative group rings, J. Algebra 34 (1975) 300–308.

[2] P.-J. Cahen, J.-L. Chabert, Coefficients et valeurs d'un polynôme, Bull. Sci. Math. 95 (2) (1971) 295–304.

[3] P.-J. Cahen, J.-L. Chabert, Integer-Valued Polynomials, American Mathematical Society Surveys and Monographs, Vol 48, American Mathematical Society, Providence RI, 1997.

[4] P.-J. Cahen, S. Gabelli, E. Houston, Mori domains of integer valued polynomials, J. Pure Appl. Algebra 153 (2000) 1–15.

[5] P.-J. Cahen, A.K. Loper, F. Tartarone, Integer-valued polynomials and Prüfer v-multiplication domains, J. Algebra 226 (2) (2000) 765–787.

[6] W. Fanggui, R.L. McCasland, On w-modules over Strong Mori domains, Comm. Algebra 25 (4) (1997) 1285–1306.

[7] W. Fanggui, R.L. McCasland, On Strong Mori domains, J. Pure Appl. Algebra 135 (2) (1999) 155–165.

[8] M. Fontana, Topologically defined classes of commutative rings, Ann. Mat. Pura Appl. IV 123 (1980) 331–355.

[9] L. Fuchs, Infinite Abelian Groups, Academic Press, New York, Vol. I, 1970, Vol. II, 1973.

[10] R. Gilmer, A two-dimensional non-Noetherian factorial ring, Proc. Amer. Math. Soc. 44 (1) (1974) 25–30.

[11] R. Gilmer, W. Heinzer, D. Lantz, The Noetherian property in rings of integer-valued polynomials, Trans. Amer. Math. Soc. 338 (1993) 187–199.

[12] P. Jaffard, Les systèmes d'ideaux, Dunod, Paris, 1960.

[13] I. Kaplansky, Commutative Rings, Revised Edition, University of Chicago Press, Chicago, London, 1974.

[14] J.R. Matijevic, Maximal ideal transforms of Noetherian rings, Proc. Amer. Math. Soc. 54 (1976) 49–52.

[15] M.H. Park, Group rings and semigroup rings over Strong Mori domains, J. Pure Appl. Algebra 163 (3) (2001) 301–318.

[16] M.H. Park, On overrings of Strong Mori domains, J. Pure Appl. Algebra 172 (1) (2002) 79–85.

[17] F. Tartarone, Divisorial prime ideals of $\mathrm{Int}(D)$ when D is a Krull-type domain, Comm. Algebra 28 (10) (2000) 4647–4663.