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A CLASSIFICATION OF THE INTEGRALLY CLOSED RINGS OF POLYNOMIALS CONTAINING $\mathbb{Z}[X]$

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ABSTRACT. We study the space of valuation overrings of $\mathbb{Z}[X]$ by ordering them using a constructive process. This is a substantial step toward classifying the integrally closed domains between $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$ that are Prüfer, the ones that are Noetherian, and the ones that are PvMDs, to name a few.

1. Introduction. We start with a brief, but substantial, preface. As stated in the abstract, the aim of this paper is to investigate the structure of the integrally closed overrings of $\mathbb{Z}[X]$ by using order-theoretic arguments on the space of the valuation overrings of $\mathbb{Z}_p[X]$ for a prime number p. The technical machinery used to reach this goal is essentially derived from MacLane's paper [19] and all the results that we use are proven in [19] for $R_P[X]$, where R is ANY Dedekind domain with finite residue fields and P is any maximal ideal. Without loss of generality with respect to MacLane's hypothesis, we chose to restrict to overrings of $\mathbb{Z}[X]$ for ease of comprehension (to balance the difficulty of the many technical aspects). But all the results given in the following can be proven exactly in the same way by replacing $\mathbb{Z}$ with any Dedekind domain with finite residue fields and considering prime elements instead of prime numbers.

Let $\mathbb{Z}$ be the ordinary ring of rational integers and let p be a fixed prime number. This paper began as the first step in an attempt to understand the structure of the set of Prüfer overrings of $\mathbb{Z}_p[X]$ by understanding the structure of the collection of valuation overrings of $\mathbb{Z}_p[X]$. The more particular focus was the collection of Prüfer domains

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which lie between $\mathbb{Z}_p[X]$ and $\mathbb{Q}[X]$. But as we worked, the project expanded to a description of all the integrally closed domains lying between $\mathbb{Z}_p[X]$ and $\mathbb{Q}[X]$. The method for obtaining this type of result has two stages. First we analyze domains of the type $V \cap \mathbb{Q}[X]$, that we call D_V , where V is a valuation overring of $\mathbb{Z}_p[X]$ in which p is a nonunit. Secondly, we let D be any integrally closed domain between $\mathbb{Z}_p[X]$ and $\mathbb{Q}[X]$, and let P be a prime ideal of D. We show that there exists a valuation overring V of D such that if M is the maximal ideal of V , then $D_P = (D_V)_{M\cap D_V}$ (Theorem 3.2). In this way, from the investigation into the simpler domains D_V we can derive results concerning global properties of D. For example, we can, as we set out to do, describe when D is a Prüfer domain. We can also say when D is a Noetherian domain, when it is a Prüfer v-multiplication domain (PvMD), when it is a Strong Mori domain, etc. (see Section 5). Now, every overring of $\mathbb{Z}[X]$ can be studied through its p-components, that is $D = \bigcap_p D_p$, where p ranges among the prime integers and D_p is the localization of D at the multplicative set $\mathbb{Z}(p)$. Hence $\mathbb{Z}_p[X] \subseteq D_p$ and the classification of the integrally closed polynomial overrings of $\mathbb{Z}_p[X]$, for every prime integer p, allows us to gain substantial knowledge about the structure of any integrally closed domain between $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$.

This program has two principal motivations.

The first is a special case of a theorem of Abhyankar, Eakin and Heinzer [1, Theorem 5.7].

Theorem 0.1. *Let $p \in \mathbb{Z}$ be a fixed prime integer. Let $V_1, V_2, \ldots , V_n$ be DVR overrings of $\mathbb{Z}_p[X]$ such that $V_i \cap \mathbb{Q} = \mathbb{Z}_p$ for each i. Let $D = V_1 \cap V_2 \cap \ldots \cap V_n \cap \mathbb{Q}[X]$. Then D is a Dedekind domain provided the residue field of each V_i is algebraic over the field of p elements. If one of the residue fields is not algebraic over the field of p elements, then D is a two dimensional Noetherian domain (in particular, D is not a Prüfer domain in this case).*

The difference between the two cases of this theorem are striking. Focus momentarily on the special case where $n = 1$ so that $D = V_1 \cap \mathbb{Q}[X]$. If D is a Dedekind domain, it turns out that the valuation overrings of D are precisely V_1 and the valuation overrings of $\mathbb{Q}[X]$. If D is not a Dedekind domain, there is a staggering infinite collection of valuation overrings of D other than V_1 and the valuation overrings of

$\mathbb{Q}[X]$ (see Remark 1.2). One possible way of viewing this phenomenon is that there is an ordering on the collection of valuation overrings of $\mathbb{Z}_p[X]$ in which p is a nonunit. When one of these valuation overrings V is intersected with $\mathbb{Q}[X]$ to obtain the domain D, the "new" valuation overrings of D are all those valuation overrings of $\mathbb{Z}_p[X]$ which are "greater" than V. In this setting, Theorem 0.1 becomes very intuitive if the DVR overrings which have a residue field which is algebraic over the field of order p are also those which are maximal in the ordering.

Notation. We let T_p denote the collection of valuation overrings of $\mathbb{Z}[X]$ which contain p as a nonunit and we call any of these *p-unitary.* For any domain $D \supseteq \mathbb{Z}[X]$, *$p$-unitary ideals* are proper ideals containing p. We denote by $\mathbb{F}_p$ the field of order p.

Theorem 0.1 is interesting, but it is also sharply limited. A particularly noteworthy limitation is that it does not deal with the case of an infinite number of valuation domains being intersected with $\mathbb{Q}[X]$.

The second major inspiration for this work is the ring $\mathrm{Int}(\mathbb{Z}) := \{f \in \mathbb{Q}[X] \mid f(\mathbb{Z}) \subseteq \mathbb{Z}\}$ of integer-valued polynomials over $\mathbb{Z}$. This domain does lie between $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$, is a two-dimensional Prüfer domain and has an uncountable number of valuation overrings in T_p, for each prime integer p. This uncountable collection of valuation overrings is very naturally indexed by the domain $\widehat{\mathbb{Z}_p}$ of p-adic integers [5]. It is instructive to note that p-adic integers are standardly represented as limits of finite sums of progressively larger and larger powers of p. A reasonable way to proceed then would seem to be to search for a similar incremental process to build "larger and larger" valuation domains in an effort to obtain maximal elements - and then Prüfer domains.

Above, we have given the major sources of inspiration for this work. The major tool employed is a 1936 paper by Saunders MacLane [19] in which he examines a procedure for building valuation overrings of $\mathbb{Z}_p[X]$. In his paper, MacLane begins with the p-adic valuation on $\mathbb{Q}$ and gives an incremental process, using what he calls *key polynomials* for building sequences of domains which have a natural ordering. Hence, his work fits the criteria laid out above. MacLane actually works in more generality than we do here, as we pointed out at the beginning of the Introduction. (He takes an arbitrary DVR inside a field K rather than $\mathbb{Z}_p$ as his starting point.)

We remark that MacLane's constructive process utilizing sequences of valuations has been rediscovered quite recently to approach different problems. For instance C. Favre & M. Jonsson in [9] adapt this method to construct trees of real valuations (passing through sequences of polynomials) which are centered on the ring $\mathbb{C}[[X,Y]]$ (see also [13]). One of the applications of these results is concerned with the study of singularities of curves (arising from $\mathbb{C}[[X,Y]]$), but the authors also give evidence to the fact that these constructions may be useful tools in topics distant from algebra/geometry.

Our procedure will be to use MacLane's ordering to study properties of some sub-collections of T_p (for a fixed prime integer p). We begin with the "minimal" element of T_p, $\mathbb{Z}_p(X)$, and use MacLane's incremental (and constructive) process to build collections of larger, and eventually, "maximal" members of T_p. In particular, when a given collection of valuation domains is intersected with $\mathbb{Q}[X]$ we want to know the structure of the resulting domain (when it is a Prüfer domain or a Noetherian domain, for instance).

Throughout, we will use a capital letter to indicate a valuation domain and the corresponding lowercase to indicate the valuation associated to it.

1. MacLane's paper.

In this section we give a substantial overview and some expansion of MacLane's results. The overview needs to be substantial because MacLane's paper is old and not well known, and also because we need to clarify and expand some points of his work which were either not dealt with fully (because of differring goals) or were expressed in language which is not contemporary.

MacLane's starting point is the valuation domain $\mathbb{Z}_p$ whose corresponding valuation is designated by v_p. We suppose that the valuation v_p on $\mathbb{Q}$ is is normalized so that $v_p(p) = 1$.

The first stage of extending from v_p to a valuation on $\mathbb{Q}(X)$ is accomplished by assigning a value to X. We extend v_p to $\mathbb{Q}(X)$ by designating $v_1(X) = \mu_1$, where μ_1 is some nonnegative real number. To begin we will restrict to the case where μ_1 is a nonnegative rational number. (Note: MacLane allows μ_1 to be any real number, but negative values of X lead to valuation domains which are not overrings of $\mathbb{Z}_p[X]$.) Following MacLane, we call V_1 *inductive commensurable* if μ_1 is rational and *inductive incommensurable* if μ_1 is irrational. We will always assume that μ_1 is rational unless otherwise specified. For simplicity of notation, we will often omit the word inductive and call these valuations directly "commensurable" or "incommensurable". We will write "inductive" (without any specification) to denote, indifferently, a commensurable or incommensurable valuation. Let $f(X) = a_nX^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0 \in \mathbb{Z}[X]$.

Then we define:

$$v_1(f(X)) := \min_{i=0,\cdots,n}\{v_p(a_i) + v_1(X^i)\} = \min_{i=0,\cdots,n}\{v_p(a_i) + i\mu_1\}$$

With this definition, v_1 can be extended to a valuation on $\mathbb{Q}(X)$. The corresponding DVR (which we designate by V_1) is known as a *first stage inductive valuation domain*. It is noteworthy that through all of MacLane's (long) paper he only rarely deals with valuation domains. His goal is to construct valuations rather than valuation domains. Hence, when he does need to work with a domain, it is sufficient for his purposes to work with subrings of the valuation domains that are associated with the valuations in question. In particular, he almost always deals with polynomials rather than rational functions. The domain $D_1 = V_1 \cap \mathbb{Q}[X]$ (which MacLane calls the value ring of $\mathbb{Q}[X]$) is the domain in which we will be principally interested. We call D_1 a *first stage inductive polynomial domain*. The maximal ideal of V_1 contracts to a height-one prime ideal in D_1 (observe that D_1 is a Krull domain, that V_1 is one of the DVR's involved in the locally finite intersection representation, and that V_1 is the only such DVR in which p is a nonunit.) Let P_1 be this height-one prime. We refer to P_1 as the *valuation prime* of D_1. We next give a result concerning overrings which is not in MacLane, but which (in a more general setting to come soon) is crucial to our main results.

Lemma 1.1. *Let α and β be two positive rational numbers with $\alpha > \beta$. Using the terminology developed above we construct two distinct first stage inductive polynomial domains $D_{1,\alpha}$ and $D_{1,\beta}$ by first setting $v_{1,\alpha}(X) = \alpha$ and then setting $v_{1,\beta}(X) = \beta$. Then $D_{1,\alpha}$ is an overring of $D_{1,\beta}$.*

Proof. It is enough to observe that $D_{1,\alpha} = \{f/p^t \mid v_{1,\alpha}(f) \geq t\}$ and that $D_{1,\beta} = \{f/p^t \mid v_{1,\beta}(f) \geq t\}$. The result follows easily from the definition of $v_{1,\alpha}$ and $v_{1,\beta}$. □

Remark 1.2. The interesting element of this result is that containment is not present when one looks at the valuation domains, but appears when one restricts to the smaller inductive polynomial domains. In fact, the containment of $D_{1,\beta}$ in $D_{1,\alpha}$ is actually important because it is a stepping stone to the observations that the commensurable valuation domain $V_{1,\alpha}$ is a localization of $D_{1,\alpha}$ (localize $D_{1,\alpha}$ at the valuation prime) and hence, $V_{1,\alpha}$ is an overring of $D_{1,\beta}$.

This fact implies that a first stage inductive polynomial domain D_1 does not fit the Dedekind case of Theorem 0.1 since there are obviously other valuation overrings of D_1 besides V_1 and the valuation overrings of $\mathbb{Q}[X]$. Hence, D_1 is a two-dimensional Noetherian domain. It follows from Theorem 0.1 that the residue field of a first stage commensurable valuation domain V_1 is not algebraic over $\mathbb{F}_p$.

We now digress momentarily to give an important structural theorem concerning domains of the type $V \cap \mathbb{Q}[X]$.

Lemma 1.3. *Let V be a valuation domain in T_p and let M be the maximal ideal of V. Let $D = V \cap \mathbb{Q}[X]$ and let $P = M \cap D$. Then:*

(1) *The radical of the principal ideal* (p) *of D is a prime ideal of D.*

(2) *If the value group of V is contained in the additive group of rational numbers, then $D_P = V$.*

Proof. (1) Note that p is a unit in $\mathbb{Q}[X]$. Let J be the radical in V of the principal ideal pV of V. Then J is a prime ideal and the contraction of J to D is the radical of pD in D.

(2) Let v be the valuation associated with V and assume that the value group of v is a subgroup of the additive group of rational numbers. Clearly, $D_P \subseteq V$. So, let f/g be a nonzero element of V with $f, g \in \mathbb{Z}[X]$. We need to show that $f/g \in D_P$. If g is a unit in V, the result is obvious. So without loss of generality, $v(f)$ and $v(g)$ are both positive. Since, the value group of V is rational and $v(p) > 0$, we

can find positive integers t, n, m, r such that $v(f^t/p^n) = v(g^m/p^r) = 0$. Then it follows that $v(f^{tr}/p^{nr}) = v(g^{mn}/p^{nr}) = 0$. Since $f/g \in V$ we must have $v(f) \geq v(g)$. It follows that we must have $mn \geq tr$. Hence, $v(f^{mn}/p^{nr}) \geq 0$. Also note that g^{mn}/p^{nr} is a unit in V and hence not in P. Since it clearly is in D we have shown that $(f/g)^{mn} \in D_P$. Since D_P is integrally closed this proves that $f/g \in D_P$. $\square$

We next explore the structure of residue fields and rings in more depth.

Proposition 1.4. ([19, Lemma 10.1 and Theorem 10.2]) *Let V_1 be a first stage inductive valuation domain and D_1 the corresponding first stage inductive polynomial domain with valuation prime P_1. Then the residue field of V_1 is the field of rational functions $\mathbb{F}_p(Y)$ in the indeterminate Y over the field $\mathbb{F}_p$. The residue ring D_1/P_1 is the ring $\mathbb{F}_p[Y]$.* $\square$

Corollary 1.5. *The height-two prime ideals of a first stage inductive polynomial domain D_1 correspond to the nonzero prime ideals of the ring $\mathbb{F}_p[Y]$.* $\square$

Proof. From Lemma 1.3, the radical of p in D_1 is P_1. A height-two prime ideal M in D_1 must contain p. In fact, if $\mathfrak{m} := M \cap \mathbb{Z}_p[X] = (0)$, then $\mathbb{Z}_p[X]_{\mathfrak{m}}$ and $(D_1)_M$ are localizations of $\mathbb{Q}[X]$, whence M is height-one. So we get the correspondence between height-two primes in D_1 and maximal ideals of $F[Y]$. $\square$

The two preceding results give the structure of the residue field (ring) of the first stage inductive valuation (polynomial) domain, but we need to examine more closely the interplay between the domain and the corresponding residue field/ring.

First, note that if we set $v_1(X) = \mu_1 = 0$ in the construction of V_1, then V_1 is the trivial extension of $\mathbb{Z}_p$ to $\mathbb{Q}(X)$, that is $\mathbb{Z}_p(X)$. In this case, it is easy to see that $D_1 = \mathbb{Z}_p[X]$ and that the valuation prime P_1 is the prime ideal generated by p. Then it is obvious that D_1/P_1 is isomorphic to $\mathbb{F}_p[Y]$. In fact, X corresponds to the variable Y in the residue ring $\mathbb{F}_p[Y]$.

Secondly, we consider the case where $\mu_1 > 0$. We cannot have the direct correspondence between X in D_1 and Y in $D_1/P_1 = \mathbb{F}_p[Y]$ that we had above because now $X \in P_1$. Since μ_1 is rational, we can find positive integers r, t such that $v_1(X^r/p^t) = 0$. If we choose r to be minimal, then X^r/p^t corresponds to Y. It should be noted here that when we extend from the first stage to the k^{th}-stage, the analogous process of finding the polynomial that "corresponds to Y" is harder.

Lemma 1.6. *With the same hypotheses and notation as Proposition* 1.4*, there exist positive integers a, b such that $Y \in \mathbb{F}_p[Y]$ corresponds to $X^a/p^b \in D_1$.* $\square$

We remarked above that in the special case where $\mu_1 = 0$, we have $D_1 = \mathbb{Z}_p[X]$. In this case, X is not in the valuation prime P_1 and the familiar height-two primes of D_1 obviously correspond exactly to the height-two primes of $\mathbb{Z}_p[X]$. In the case where $\mu_1 > 0$, it does happen that $X \in P_1$. By construction, D_1 is a Krull domain and P_1 is the only height-one prime of D_1 which contains p. Any height-two prime of D_1 must contain p (if not it is an upper to zero and it is height-one). It follows that each height-two prime of D_1 contains both X and p and so contains the prime ideal generated by X, p in $\mathbb{Z}_p[X]$. This is not a profound result in itself, but is good to keep in mind for the sake of intuition as we expand to a more general situation.

There are two more points to be made concerning the first stage situation before we press on to the k^{th}-stage.

First, we observe that the finite field that appeared in the residue rings/fields was always the field of p elements. This is an anomaly resulting from the fact that X is an irreducible polynomial of degree 1. We will soon encounter larger finite fields as a result of dealing with higher degree polynomials.

Second, note that p generates the maximal ideal of $\mathbb{Z}_p$, but p does not necessarily generate the maximal ideal of V_1. Since we are following MacLane in focussing more on the commensurable domain of polynomials D_1, we deal with the equivalent issue of what the smallest value of $v_1(f)$ is for f in the valuation prime $P_1 \subseteq D_1$. If μ_1 is an integer, then $v_1(p) = 1$ is minimal. If μ_1 is not an integer, then there exist polynomials $f \in P_1$ with $v_1(f) < 1$. Since we assumed μ_1 to be rational, the minimum does always exist and it is easily seen to always be of the form $1/e$ for some positive integer e. (In fact, e is the exponent a of Lemma 1.6.) In this case, an element of minimal value in P_1 is any element of valuation $1/e$.

We now depart from the first stage setting and deal with k^{th}-stage inductive valuations. In particular, we show how to construct $k+1^{th}$-stage inductive valuations from the k^{th}-stage inductive ones.

Assume that k is a positive integer and that we have constructed k^{th}-stage inductive (commensurable) domains (valuation and polynomial) with the following properties.

(1) The k^{th}-stage inductive (commensurable) valuation domain V_k is a DVR with residue field isomorphic to the field $F_k(Y)$ of rational functions in the variable Y over the finite field F_k.

(2) The k^{th}-stage inductive polynomial domain $D_k = V_k \cap \mathbb{Q}[X]$ is a two-dimensional Krull domain.

(3) The maximal ideal of V_k contracts to the valuation prime P_k of D_k.

(4) The residue ring D_k/P_k is isomorphic to the ring $F_k[Y]$ of polynomials in Y over the finite field F_k. The height-two primes of D_k correspond to the irreducible polynomials of $F_k[Y]$.

The key polynomial. The process of going from the k^{th}-stage to the $k+1^{th}$-stage involves a monic, irreducible polynomial $\phi_{k+1}(X) \in P_k$ which is called a *key polynomial*. In the first stage part of this section we used the first key polynomial $\phi_1(X) = X$ to progress from $\mathbb{Z}_p$ to V_1. MacLane uses a property he calls *equivalence irreducibility* to identify which polynomials are eligible to be key polynomials. We will not discuss this property but use, instead, one of several later characterizations given by MacLane to identify the potential key polynomials [19, Theorem 13.1].

First, we introduce some new terminology. Let v_k be the valuation associated with V_k (normalized so that $v_k(p) = 1$). Let $f(X) \in P_k$. Choose positive integers r, t such that $v_k(f(X)^r/p^t) = 0$. We know that $f(X)^r/p^t$ corresponds to some nonzero element of $F_k[Y]$. It may be a nontrivial polynomial in Y or it may be a unit in F_k. If $f(X)^r/p^t$ corresponds to a unit in F_k, we say that $f(X)$ is a *residue-unit* of D_k.

(Note that there is more than one possible choice of the integers r, t. The residue-unit property is independent of which choice is made.)

Choose an irreducible polynomial $\Psi(Y) \in F_k[Y]$ (with the single exception that $\Psi(Y) \neq Y$). Let H_k be the canonical homomorphism from D_k onto $F_k[Y]$. Intuitively, our goal is to find an irreducible polynomial $f(X) \in P_k$ such that $H_k(f(X)) = \Psi(Y)$. This is impossible since $f(X) \in P_k$ implies $H_k(f(X)) = 0$. Using the first stage procedures as a model, we could look for a polynomial of the form $f(X)/p^t$ with $f(X) \in P_k$, $v_k(f(X)/p^t) = 0$ and $H_k(f(X)/p^t) = \Psi(Y)$. Such an $f(X)$ can be found, but will likely not be irreducible. What we can do is find an irreducible polynomial $\phi(X) \in P_k$ and a residue-unit polynomial $f(X) \in P_k$ such that the following hold:

- for some positive integer t we have: $v_k(f(X)/p^t) = -v_k(\phi(X))$;
- $H_k(\phi(X)f(X)/p^t) = \Psi(Y)$.

Notes:

(1) MacLane designates the polynomial $f(X)/p^t$ as $R(X)$.

(2) The restriction that $\Psi(Y) \neq Y$ will, at a later point, be eliminated. This restriction prevents some confusing redundancy and makes theorem statements cleaner. Nonetheless, this will present some problems for us which were not a concern for MacLane. In some cases we have an inductive (commensurable) domain V and we would like to extend V using a key polynomial arising from $\Psi(Y) = Y$. In such cases we think of V as being an extension of an inductive (commensurable) domain W using a key polynomial $\phi(X)$ with the assignment $v(\phi(X)) = \mu > w(\phi(X))$. We can then work toward the desired extension of V by extending W using the same key polynomial $\phi(X)$ but assigning it a larger value than μ. This will suffice to deal with extensions involving $\Psi(Y) = Y$.

Choose $\phi(X)$ as above so that the degree is minimized and that $\phi(X)$ is monic. Then $\phi(X)$ is a possible choice for a key polynomial $\phi_{k+1}(X)$ to extend from the k^{th}-stage to the $k + 1^{th}$-stage.

More precisely we have the following result.

Proposition 1.7. [19, Theorem 13.1] *For any monic, irreducible polynomial $\Psi(Y) \in F_k[Y]$ (except $\Psi(Y) \neq Y$) we can find a corresponding key polynomial $\phi_{k+1}(X)$ (monic, irreducible, and of minimal degree) such that $H_k(\phi_{k+1}(X)f(X)/p^t) = \Psi(Y)$, where t is a positive integer and $f \in P_k$ (with f a residue unit).* □

It should be noted that, given a choice of $\Psi(Y)$, the choice of $\phi_{k+1}(X)$ is not unique. We do have something close to uniqueness however. Let V be an inductive domain and choose a polynomial $\Psi(Y)$. Let $\phi_1(X)$ and $\phi_2(X)$ be two possible key polynomials corresponding to $\Psi(Y)$ and extend V to another inductive domain W using $\phi_1(X)$. Then it is possible to use $\phi_2(X)$ to construct a proper extension V_2 such that W is either equal to V_2 or is a proper extension of V_2. The import of this is that we lose nothing by focussing exclusively on one key polynomial corresponding to $\Psi(Y)$.

Extension procedure from the k^{th}-stage to the $k+1^{th}$-stage. See ([19, Section 4]. Assume that we have chosen a key polynomial $\phi_{k+1}(X)$. Choose a positive rational number μ_{k+1} so that $\mu_{k+1} > v_k(\phi_{k+1}(X))$. We define $v_{k+1}(\phi_{k+1}(X)) = \mu_{k+1}$. Also let $v_{k+1}(p) = 1$. Then let $h(X)$ be any nonzero polynomial in $\mathbb{Q}[X]$. We define $v_{k+1}(h(X))$ by expanding $h(X)$ in powers of $\phi_{k+1}(X)$. In other words, we divide $h(X)$ by the highest possible power of $\phi_{k+1}(X)$, divide the remainder by the highest possible power of $\phi_{k+1}(X)$, and so on, until we obtain an expansion:

$$\begin{aligned} h(X) = a_n(X)(\phi_{k+1}(X))^n + a_{n-1}(X)(\phi_{k+1}(X))^{n-1} + \dots \\ + a_1(X)(\phi_{k+1}(X)) + a_0(X) \end{aligned}$$

with $\deg(a_i(X)) < \deg(\phi_{k+1}(X))$, for each $i = 0, \cdots, n$. We then define:

$$v_{k+1}(h(X)) := \min_{i=1,\cdots,n}\{v_k(a_i(X)) + i\mu_{k+1}\}.$$

This extends naturally to a valuation v_{k+1} on $\mathbb{Q}(X)$.

Now we can define the $k+1^{th}$-stage inductive valuation domain V_{k+1} associated with v_{k+1} and the $k+1^{th}$-stage inductive polynomial domain

$D_{k+1} = V_{k+1} \cap \mathbb{Q}[X]$. As before, the maximal ideal of V_{k+1} contracts to the height-one prime P_{k+1} of D_{k+1}, which we call the valuation prime of D_{k+1}. More generally, properties 1-4 (listed at page 8) which we assumed about the k^{th}-stage domains, also hold for the $k+1^{th}$-stage domains, i.e.

(1) The $k+1^{th}$-stage inductive valuation domain V_{k+1} is a DVR with residue field isomorphic to the field $F_{k+1}(Y)$ of rational functions in the variable Y over the finite field F_{k+1}.

(2) The $k+1^{th}$-stage inductive polynomial domain $D_{k+1} = V_{k+1} \cap \mathbb{Q}[X]$ is a two-dimensional Krull domain. Moreover, D_{k+1} is Noetherian by Theorem 0.1.

(3) The maximal ideal of V_{k+1} contracts to the valuation prime P_{k+1} of D_{k+1}.

(4) The residue ring D_{k+1}/P_{k+1} is isomorphic to the ring $F_{k+1}[Y]$ of polynomials in Y over the finite field F_{k+1}. The height-two primes of D_{k+1} correspond to the irreducible polynomials of $F_{k+1}[Y]$.

Note that the structure of the residue ring D_{k+1}/P_{k+1} leads to the conclusion that the maximal ideals of $F_{k+1}[Y]$ correspond to height-two prime ideals of D_{k+1}. It is very evident that each maximal ideal of $F_{k+1}[Y]$ is the image under H_{k+1} of a height-two prime of D_{k+1} (containing P_{k+1}). It is less evident, perhaps that these are the only height-two primes of D_{k+1}. From the proof of Lemma 1.3, we know that the radical of p in D_{k+1} is the contraction of the radical of p in V_{k+1}, and this is the maximal ideal of V_{k+1}. Hence, P_{k+1} is the radical of p.

A height-two prime in an overring of $\mathbb{Z}_p[X]$ must contain p (if $\mathfrak{m}$ is a prime ideal not containing p, then $\mathbb{Z}_p[X]_{\mathfrak{m}}$ is a localization of $\mathbb{Q}[X]$, whence $\mathfrak{m}$ is height-one), so it must contain P_{k+1}. Hence, the claimed correspondence between height-two primes in D_{k+1} and maximal ideals of $F_{k+1}[Y]$ holds.

Monotonicity of the values on polynomials. There is more to be said about residue rings, but we first need to comment on the comparisons that can be made between the valuations v_k and v_{k+1}. MacLane proves that with the given definitions, for a nonzero polynomial $f \in \mathbb{Q}[X]$ we have $v_{k+1}(f) \geq v_k(f)$ [19, Theorem 5.1] and the inequality sign holds if and only if f is *equivalence-divisible* by ϕ_{k+1} in V_k. We will neglect discussion of *equivalence-divisibility* and analyze anew the difference between " $>$ " and " $\geq$ " in the comparison of v_k and v_{k+1}. As noted before, the principal difficulty we have is that MacLane was interested in valuations and not in ideals. We will examine the ideal-theoretic implications of MacLane's monotonicity results.

As before, let

$$f = a_n\phi_{k+1}^n + a_{n-1}\phi_{k+1}^{n-1} + \ldots + a_1\phi_{k+1} + a_0,$$

with $\deg(a_i) < \deg(\phi_{k+1})$, for each $i = 0, \cdots, n$, and

$$v_{k+1}(f) := \min_{i=0,\cdots,n}\{v_k(a_i) + i\mu_{k+1}\}.$$

To begin, we note that:

Lemma 1.8. *With the above hypotheses and notation $v_{k+1}(f) \geq v_k(f)$, for all $f \in \mathbb{Q}[X]$. Inequality holds if and only if $v_k(a_0) > v_k(f)$.*

Proof. Maclane proves that $v_{k+1}(f) \geq v_k(f)$ for all $f \in \mathbb{Q}[X]$. It follows easily that $v_{k+1}(f) = v_k(a_0) = v_k(f)$ if $v_k(a_0) = v_k(f)$. Suppose that $v_k(a_0) > v_k(f)$ and write $f = \phi_{k+1}g + a_0$. The fact that $v_{k+1}(\phi_{k+1}) > v_k(\phi_{k+1})$ then finishes the proof. □

Recall that the maximal ideal of the DVR V_k contracts to the height-one prime P_k of the polynomial domain D_k. Hence, elements of height-two primes of D_k which lie outside of P_k have value 0. The key polynomial ϕ_{k+1} of the discussion above lies in P_k and hence $v_k(\phi_{k+1}) > 0$. However, we can find positive integers r, t so that $v_k(\phi_{k+1}^r/p^t) = 0$. We also note that $v_{k+1}(\phi_{k+1}^r/p^t) > 0$. So there are elements of D_k (including some with 0 value in V_k) which increase in value when passing from v_k to v_{k+1}. In fact, we can say something about the structure of this set of objects combined with P_k.

Lemma 1.9. *Let $L_{\Psi,k} = \{d \in D_k | v_{k+1}(d) > v_k(d)\}$. Then $M_{\Psi,k} := L_{\Psi,k} \cup P_k$ is a prime ideal of D_k.*

Proof. Straightforward. □

In fact, we can say a great deal more about this prime ideal. Note that $\phi_{k+1}^r/p^t \in M_{\Psi,k}$. The following result is then immediate.

Lemma 1.10. $H_k(M_{\Psi,k}) = (\Psi(Y))F_k[Y]$.

Now take note of the description of $M_{\Psi,k}$ as the union of the elements of P_k (which have positive value under v_k) and the elements of D_k which have 0 value under v_k but positive value under v_{k+1}. Then recall that P_{k+1} is the set of elements of D_{k+1} which have positive value under v_{k+1}. The next result follows immediately, since every height 2 prime ideal of D_{k+1} contains P_{k+1}.

Lemma 1.11. $P_{k+1} \cap D_k = M_{\Psi,k}$. *In fact, since* $M_{\Psi,k}$ *is maximal it follows that if* M *is any height-two prime of* D_{k+1}*, then* $M \cap D_k = M_{\Psi,k}$.

Now suppose that we are given a commensurable valuation domain V_k. To extend V_k to another inductive domain we choose a key polynomial ϕ_{k+1}. We then learn that when we extend V_k using ϕ_{k+1} we get a new inductive valuation domain V_{k+1}. The valuation prime of V_{k+1} will lie over a maximal ideal M_Ψ of $D_k = V_k \cap \mathbb{Q}[X]$: M_Ψ corresponds to an irreducible polynomial $\Psi(Y) \neq Y$ in the residue field of V_k and $\Psi(Y)$ also corresponds to ϕ_{k+1}. (We could have begun with either M_Ψ or $\Psi(Y) \neq Y$ and then chosen ϕ_{k+1} if we had desired.) This summarizes the process we have dealt with thus far.

We could, conversely, be given a commensurable domain V_k and another inductive domain W which we are told is an n^{th}-stage extension of V_k, for some $n > 0$. Suppose the maximal ideal of W lies over the maximal ideal M_Ψ of the polynomial domain $D_k = V_k \cap \mathbb{Q}[X]$, with M_Ψ corresponding to a polynomial $\Psi(Y) \neq Y$ in the residue field of V_k. We can then choose a potential key polynomial ϕ_{k+1} in D corresponding to $\Psi(Y)$. Then, since the maximal ideal of W lies over M_Ψ, we know that $w(\phi_{k+1}) > v(\phi_{k+1})$. So we can extend V by using the key polynomial ϕ_{k+1} and assigning to it the value $w(\phi_{k+1})$. This yields a first-stage extension of V_k, V_{k+1}. Either V_{k+1} is equal to W or W is a proper extension of V_{k+1}. In the latter case we simply repeat the process (for details, see Proposition 2.5).

In both of the above paragraphs, the possibility that $\Psi(Y) = Y$ is not dealt with. It surely can happen though that we could have a commensurable domain V_k and another inductive domain W such that the maximal ideal of W lies over the maximal ideal M_Y of $D_k = V_k \cap \mathbb{Q}[X]$ corresponding to the polynomial $\Psi(Y) = Y$ of the residue field of V_k. In fact it is easy to obtain such an extension. Let V_{k-1} be an inductive commensurable domain such that V_k can be obtained from V_{k-1} using the key polynomial ϕ_k by assigning the new value $\mu_k = v_k(\phi_k)$. In this case, we can extend V_k to a new inductive domain W by beginning with V_{k-1}, using the same key polynomial ϕ_k, and assigning a new value $w(\phi) = \mu_w > \mu_k$. It is easy to see that this extension behaves very much like the extensions considered before. In particular, the maximal ideal of W does lie over the maximal ideal M_Y of D_k and the set of elements of D_k which increase in value when we extend to W is exactly the set $L_{\Psi,k} = \{p^t f | t \geq 0, f \in M_{\Psi,k} - P_k\} = \{f \in D_k | \Psi(Y) \mid H_k(f), H_k(f) \neq 0\}$.

Conversely, suppose we are simply given an inductive commensurable domain V_k and an extension W such that the maximal ideal of W lies over the maximal ideal M_Y of $D_k = V_k \cap \mathbb{Q}[X]$. Again we retreat to an inductive domain V_{k-1} such that V_k is an extension of V_{k-1} using the key polynomial ϕ_k. Since the maximal ideal of W lies over M_Y it is easy to see that $v_{k-1}(\phi_k) < v_k(\phi_k) < w(\phi_k)$. So we can extend V_k in the direction of W by using the key polynomial ϕ_k for V_{k-1} and assigning it the value $w(\phi_k)$.

We finish our discussion of inductive commensurable domains with a simple but important lemma.

Lemma 1.12. [19, (6) p. 387] *For each irreducible polynomial $\Psi(Y) \in F_{k+1}[Y]$ (we can include $\Psi(Y) = Y$ here) there is a unique maximal ideal M_Ψ of D_{k+1} which contains an element d such that $H_{k+1}(d) = \Psi(Y)$. Also, the inductive construction allows us to assume that the polynomial in D_{k+1} which corresponds to Y in this residue ring can be taken to have the form $Q(X)\phi_{k+1}^{\tau_{k+1}}(X)$, where $Q(X) = f(X)/p^t$ for some residue-unit $f \in P_{k+1}$ and a positive integer τ_{k+1}.*

MacLane's stated goal is "given all such values for the field R of rational numbers, we construct all possible values of the ring $R[X]$ of all polynomials in X with coefficients in R."

In contemporary terms, his goal was to construct all 1-dimensional valuation overrings of $\mathbb{Z}[X]$ (or more precisely, the corresponding valuations). He accomplished this goal. His methods also produce many two-dimensional valuation overrings and many one-dimensional valuation overrings which are not included in the class of inductive domains we have dealt with so far. There is also a class of two-dimensional valuation overrings which can be obtained from MacLane's domains but not with his methods. We turn now to examining those valuation domains other than the inductive commensurable domains we have discussed thus far.

Inductive incommensurable valuations. Let V_k be a k^{th}-stage commensurable valuation domain. Also let ϕ_{k+1} be a key polynomial for V_k. As before we extend V_k to a new valuation domain V_{k+1} using the key polynomial ϕ_{k+1}. Everything is as before except that now $\mu_{k+1} = v_{k+1}(\phi_{k+1})$ is an irrational number. We refer to the valuation domain V_{k+1} as an incommensurable $k+1^{th}$-stage inductive domain. Let $D_{k+1} = V_{k+1} \cap \mathbb{Q}[X]$ be the corresponding polynomial domain and let P_{k+1} be the valuation prime (the contraction of the maximal ideal of V_{k+1} to D_{k+1}).

MacLane demonstrates that this incommensurable V_{k+1} behaves differently than the commensurable valuation domains do.

Proposition 1.13. ([19, Theorem 14.2]) *Let V_{k+1} be an inductive incommensurable domain as described above. Then the residue field of V_{k+1} is a finite field F_{k+1}.*

MacLane does not discuss the residue ring D_{k+1}/P_{k+1} as he does with the commensurable case, but the structure is easy to deduce.

Corollary 1.14. *Let D_{k+1} and P_{k+1} be as above. Then $D_{k+1}/P_{k+1} = F_{k+1}$. In particular, P_{k+1} is a maximal ideal.*

In fact, P_{k+1} is the unique height-two prime of D_{k+1}. We prove this in several stages.

Lemma 1.15. *P_{k+1} is a height-two prime ideal.*

Proof. Consider the polynomial $\phi = \phi_{k+1}$. We know by construction that ϕ is an irreducible polynomial in $\mathbb{Q}[X]$. Consider the prime ideal $P_\phi = (\phi)\mathbb{Q}[X] \cap D_{k+1}$. Certainly this is a nonzero prime ideal of D_{k+1}. Let $(\phi f)/p^t \in P_\phi$. Recall from the discussion preceding Lemma 1.8 that $v_{k+1}(f) = v_k(a_i) + i\mu_{k+1}$ where $v_k(a_i)$ is rational and i is nonnegative. Hence, $v_{k+1}((\phi f)/p^t) = (i+1)\mu_{k+1} + a_i - t$. Then, $v_{k+1}((\phi f)/p^t)$ is not zero since μ_{k+1} is irrational. Thus, $(\phi f)/p^t \in P_{k+1}$ and so $P_\phi \subseteq P_{k+1}$. Since $p \in P_{k+1}$ and $p \notin P_\phi$ this proves that P_{k+1} is a height-two prime. □

Lemma 1.16. *P_{k+1} is the unique height-two prime of D_{k+1} (and is the radical of p).*

Proof. First we observe that, as noted before, any prime of D_{k+1} which does not contain p must have height one since localization would yield a valuation overring of $\mathbb{Q}[X]$. Hence, any height-two prime of D_{k+1} must contain p. Also note that the maximal ideal of V_{k+1} is the radical of pV_{k+1} and so Lemma 1.3 demonstrates that P_{k+1} is the radical of p. The result follows. □

Corollary 1.17. *If D_{k+1} is an incommensurable polynomial domain then it is not Noetherian.*

Corollary 1.18. *If D_{k+1} is an incommensurable polynomial domain then $(D_{k+1})_{P_{k+1}} \neq V_{k+1}$.*

Note that the preceding two results contrast sharply with the situation for the inductive commensurable domains.

Limit valuations. Consider a sequence of inductive valuations $\{V_k\}_{k\geq 0}$, that is, V_k is a first stage extension of V_{k-1} according to MacLane's procedure. From the monotonic properties we know that $v_{k+1}(f) \geq v_k(f)$ for each nonzero $f \in \mathbb{Q}[X]$. Then we define $v_\infty(f) := \lim_{k\to\infty} v_k(f)$. MacLane proves ([19, Theorem 6.2]) that the function defined by v_∞ on $\mathbb{Q}[X]$ extends to a valuation on $\mathbb{Q}(X)$. We say V_∞ is the valuation domain associated to v_∞ and we write $V_\infty = \lim_{k\to\infty} V_k$ to specify that it is the limit of the sequence $\{V_k\}_{k\geq 0}$. V_∞ will be referred to as a *limit valuation domain*.

From the definition of limit valuation the following lemma follows easily:

Lemma 1.19. *Let V be a limit valuation domain defined by a sequence of inductive valuation domains $\{V_k\}_{k\geq 0}$. Then:*

$$V \cap \mathbb{Q}[X] = \bigcup_{k\geq 0} V_k \cap \mathbb{Q}[X].$$

Note: It should be noted that $v_\infty(f) = \infty$ is possible for a nonzero polynomial f. MacLane makes note of this and deals with it very lightly. If v_∞ takes finite values for every nonzero polynomial, we follow MacLane and refer to V_∞ as a finite limit valuation domain. If V_∞ is a limit valuation domain which is not finite we refer to it as an infinite limit valuation domain. We also observe that a finite limit valuation domain is one-dimensional and an infinite limit valuation domain is two-dimensional. We deal with the finite case first and then examine the infinite case.

Proposition 1.20. [19, Theorem 14.1] *Suppose that V_∞ is a finite limit valuation domain. Then the residue field of V_∞ is algebraic over the field $\mathbb{F}_p$ of p elements. (Note that the residue field may be infinite.)*

As with the inductive domains we define $D_\infty := V_\infty \cap \mathbb{Q}[X]$ and call D_∞ a finite limit polynomial domain. And we again designate P_∞ to be the valuation prime (the contraction of the maximal ideal of V_∞ to D_∞). The following corollary is then immediate.

Corollary 1.21. *Suppose that V_∞ is a finite limit valuation domain. Then P_∞ is a maximal ideal of D_∞.*

In fact, much more is true. From Lemma 1.3 we know that P_∞ is the radical of p (since the radical of p in V_∞ is the maximal ideal) and, since P_∞ is maximal, P_∞ is the only prime to contain p. Lemma 1.3 then also implies that $(D_\infty)_{P_\infty} = V_\infty$. It is easy to see that $(D_\infty)_P$ is a valuation domain for each P which does not contain p (in this case P is an upper to zero and D_P is a localization of $\mathbb{Q}[X]$). So we have proven the following result.

Corollary 1.22. *Suppose that V_∞ is a finite limit valuation domain. Then $D_\infty = V_\infty \cap \mathbb{Q}[X]$ is a one-dimensional Prüfer domain with V_∞ and the valuation overrings of $\mathbb{Q}[X]$ as its valuation overrings.*

We now make a few observations regarding infinite limit valuation domains.

Lemma 1.23. *Let V_∞ be an infinite limit valuation domain. Then all of the polynomials with infinite value are multiples of one single irreducible polynomial f.*

Proof. Suppose that f is an irreducible polynomial such that $\lim_{k\to\infty} v_k(f) = \infty$. Let g be another polynomial such that $\lim_{k\to\infty} v_k(g) = \infty$ and suppose that g is not a multiple of f. Then $af + bg = 1$ for some rational numbers a and b. Since $v_k(a) = v_{k+1}(a)$ and $v_k(b) = v_{k+1}(b)$ for all values of k then $\lim_{k\to\infty} v_k(1) = \infty$ which is a contradiction. $\square$

MacLane treats this infinite case essentially as a one-dimensional valuation domain with the value group extended by adding ∞. In fact, the valuation domain in question is two-dimensional and it is easy to see that the one-dimensional overring is the domain $\mathbb{Q}[X]_{(f)}$.

Conversely one can begin with an irreducible polynomial $f \in \mathbb{Q}[X]$ and build a two-dimensional valuation domain with $\mathbb{Q}[X]_{(f)}$ as the one-dimensional overring. Such a domain can be constructed by beginning with the domain $\mathbb{Q}[X]_{(f)}$ and using a standard pullback construction to obtain the two-dimensional valuation subring.

This leads naturally to the question of whether all valuation domains constructed in this manner can be obtained with MacLane's construction.

The answer is that all but one such valuation domain can be obtained using MacLane's methods. This can be demonstrated by making minor modifications to the proof of [19, Theorem 8.1]. The key to this proof is to start with the valuation W that you wish to represent as a limit of inductive valuations and build an increasing sequence of inductive valuations $\{V_k\}_{k\geq 0}$ that are all smaller than W, but have increasingly larger and larger bases of agreement with it. The only difficulty in applying MacLane's proof directly is that we build the sequence of V_k's by choosing potential key polynomials and assigning to them the appropriate value to match the value in W. This would be problematic if the polynomial that we chose was the polynomial f which is to have infinite value in W. If we do land on this choice, we resolve the problem by adding a large power of p to f. This will give an alternate choice for the key polynomial. And the proof demonstrates that with this process any polynomial which does not eventually get assigned a W-compatible value will be a polynomial which has infinite value, which we have already demonstrated can only be f.

The one valuation domain which cannot be obtained by MacLane's methods is the unique valuation domain for which the polynomial $f(X) = X$ has infinite value. The problem is that stage one of MacLane's method involves assigning a real number value to X. This can be worked around in two different ways. We can assign ∞ as the value of X at the first stage. We can also go back to the start of MacLane's inductive process and let $V_{n,1}$ be the first stage inductive domain obtained by assigning the value $v_{n,1}(X) = n$, for each positive integer n. We then let V_∞ be the limit of the sequence $\{V_{n,1}\}_{n\geq 0}$ in the same manner as MacLane constructed limit values.

For all infinite limit valuation domains we can state some results and define some terms as we did with the previous classes of domains.

Proposition 1.24. *Suppose that V_∞ is an infinite limit valuation domain. Then the residue field is a finite field of order p^n for some $n > 0$.*

Proof. Recall that the one dimensional valuation overring of V_∞ is of the form $\mathbb{Q}[X]_{f(X)}$ for some irreducible polynomial $f(X) \in \mathbb{Q}[X]$. It follows that the residue field of V_∞ is the same as a residue field of the ring of integers of a finite degree extension of $\mathbb{Q}$. Since p is a nonunit in V_∞ the result follows. □

As with the inductive domains and the finite limit domains we define $D_\infty := V_\infty \cap \mathbb{Q}[X]$ and call D_∞ an infinite limit polynomial domain. And we again designate P_∞ to be the valuation prime (the contraction of the maximal ideal of V_∞ to D_∞). From Lemma 1.23, it follows that p has finite value, whence the radical of p in V_∞ is the maximal ideal and, by Lemma 1.3, P_∞ is the radical of p in D_∞. The following corollary is then immediate.

Corollary 1.25. *P_∞ is a height-two maximal ideal of D_∞.*

In fact, as with the finite limit domains, much more is true. D_∞ is actually a two-dimensional Prüfer domain and P_∞ is the unique prime ideal containing p. This can be proven with almost the same proof as that given for the finite case. We now show that $(D_\infty)_{P_\infty} = V_\infty$. Let $f, g \in D_\infty$. If both f, g have finite value, the proof of Lemma 1.3 suffices to show that either f/g or g/f is in $(D_\infty)_{P_\infty}$. If f, g both have infinite value they must have a common factor by Lemma 1.23. This common factor can be cancelled when considering the question of whether f/g or g/f lies in $(D_\infty)_{P_\infty}$. So assume that f has infinite value and g has finite value. The proof of Lemma 1.3 can be easily adjusted to show that $f/g \in (D_\infty)_{P_\infty}$. We have proven the following result.

Corollary 1.26. *Suppose that V_∞ is an infinite limit valuation. Then D_∞ is a two-dimensional Prüfer domain with V_∞ and the valuation overrings of $\mathbb{Q}[X]$ as its valuation overrings.*

The construction of the limit valuation domains is fairly straightforward. Except for the very easily controlled infinite situation we just considered the limit valuation construction involves infinite sequences of values $\{v_k(f)\}_{k\geq 0}$ which are actually constant after some finite stage. The construction of our last class of valuation domains is not as elementary. Before we press on to this last class of domains, we briefly discuss the question of uniqueness.

Given two inductive domains, V_1, W_1 with $W_1 \cap \mathbb{Q}[X] \subseteq V_1$ and V_1 commensurable it is not the case that there is a unique chain of simple key polynomial extensions which connect V_1 and W_1. However, MacLane proves in Section 16 of his paper that any inductive or limit domain can be expressed uniquely as the end (or limit) of a "homogeneous" chain of valuation domains. The nature of homogeneous values is not of importance. The important points for us are those enumerated in the following result.

Definition 1.27.

(1) Given a valuation domain $V \in T_p$, we say that V is a *MacLane valuation domain* if V is an inductive (commensurable or uncommensurable) or a limit valuation domain in the sense explained before.

(2) Given two MacLane valuation domains $V, W \in T_p$, with V inductive commensurable, we say that W is a *MacLane extension* (or simply an *extension*) of V if W is an inductive (commensurable or incommensurable) k^{th}-stage extension of V, for some integer $k < \infty$, or if W is the limit of a sequence, containing V, of commensurable valuation domains.

Lemma 1.28. ([19, proof of Theorem 16.3]) *Let V be a limit or inductive (commensurable or incommensurable) domain. Then V can be represented as either the last member of a finite homogeneous chain or as the limit of an infinite homogeneous chain (indexed by the positive integers). The following are properties of this chain.*

(1) *The homogeneous chain begins with $V_0 = \mathbb{Z}_p$.*

(2) *The homogeneous chain representing V is unique.*

(3) *The homogeneous chain representing V is defined inductively.*

(4) *Each member of the chain (except possibly the last member) is inductive commensurable and it is a MacLane extension of the previous one.*

(5) *The chain cannot be refined.*

Now we depart momentarily from the classification of valuation domains to discuss a method of constructing valuation domains which will prove useful.

Ultrafilter limits Let $\{D_\lambda\}_{\lambda\in\Lambda}$ be a collection of integral domains with nonzero intersection, each with quotient field K. Let U be an ultrafilter on Λ. Following [6, Lemma 2.9], we define an ultrafilter limit of $\{D_\lambda\}_{\lambda\in\Lambda}$ using U. For $d \in K$ let $C(d) = \{\lambda \in \Lambda | d \in D_\lambda\}$. Then we define $D_U := \{d \in \mathbb{Q}(X) | C(d) \in U\}$ and we have the following result.

Lemma 1.29. *Let the notation be as in the previous paragraph. Then:*

(1) *D_U is an integral domain contained in K.*

(2) *If each D_λ is a valuation domain, then D_U is also a valuation domain with maximal ideal $M_U = \{d \in K | B(d) \in U\}$ where $B(d) = \{\lambda | v_\lambda(d) > 0\}$.*

(3) *If each D_λ is an overring of a domain T, then D_U is also an overring of T.*

Upside-down valuations. We recap our progress in classifying valuation domains. MacLane classified all of the one-dimensional valuation domains in T_p. These domains consist of the classes of inductive domains (commensurable and incommensurable) and the finite limit domains. Two-dimensional valuation domains certainly exist. A useful observation is that any two-dimensional valuation domain must have a one-dimensional valuation overring. There are two possibilities to consider.

- The one-dimensional valuation overring V does not contain p as a nonunit. In this case V would necessarily be of the form $\mathbb{Q}[X]_{(f)}$ for some irreducible polynomial $f \in \mathbb{Q}[X]$. These domains have already been seen to all fall into the class of infinite limit valuation domains.

- The prime p is a nonunit in the one-dimensional valuation overring V. In this case V is one of the one-dimensional domains in T_p. These domains have been classified. The classes are: inductive commensurable, inductive incommensurable, finite limit. The residue field of an inductive incommensurable or a finite limit valuation domain is algebraic over a finite field. Since these fields do not admit any nontrivial valuation it follows easily that an inductive incommensurable or finite limit valuation domain cannot be the one-dimensional valuation overring of a two-dimensional valuation domain. Hence the valuation domains left to classify must be two-dimensional valuation domains which have inductive commensurable domains as overrings.

The domains in this final class will be referred to as *upside-down* domains. They can be constructed in three distinct ways, as pullbacks of inductive commensurable domains, as ultrafilter limits of sequences of inductive commensurable domains, and as MacLane extensions of inductive commensurable domains with extended values lying outside of the real numbers. The ultrafilter and MacLane extension descriptions will be most useful in the sequel, but the pullback description is needed in order to understand what results are obtained using the other methods.

The pullback construction. Suppose that V is an upside-down valuation domain with maximal ideal M and height-one prime Q. Then V can be constructed from V_Q by means of the following pullback.

$$\begin{tikzcd} V \arrow[r] \arrow[d] & V/Q \arrow[d] \\ V_Q \arrow[r] & V_Q/QV_Q \end{tikzcd}$$

Recall that V_Q is a commensurable valuation domain. (In the following we will refer to V_Q as the commensurable valuation domain *corresponding* to the upside-down domain V.) Hence the residue field of V_Q is of the form $F(Y)$, where F is a finite extension of $\mathbb{F}_p$. Moreover, since V/Q is a valuation domain, it is of the form $F[Y]_{(f)}$, for some irreducible polynomial $f \in F[Y]$ or it is $F[1/Y]_{(1/Y)}$. It follows that the maximal ideal of V is principal since V/Q is a DVR.

For a particular example, consider the pullback:

$$\begin{tikzcd} V \arrow[r] \arrow[d] & \mathbb{F}_p[\overline{X}]_{(\overline{X})} \arrow[d] \\ \mathbb{Z}[X]_{p[X]} \arrow[r] & \mathbb{F}_p(\overline{X}) \end{tikzcd}$$

where $\overline{X}$ is the class of X modulo $p[X]$. In this case V is a valuation domain in which the maximal ideal is generated by X and the height-one prime ideal contains p.

The ultrafilter construction. As noted in Lemma 1.29, we can start with a collection of valuation domains with nonzero intersection and construct an ultrafilter limit of the collection, which will itself be a valuation domain. To realize the upside-down domains in this manner, we need to consider three separate cases. Suppose that V is an upside-down domain with Q a height-one prime and V_Q an inductive commensurable domain.

(1U) $V/Q = F[Y]_{(\Psi)}$ where $\Psi(Y) \neq Y$ is an irreducible polynomial in $F[Y]$. Consider the inductive commensurable valuation domain V_Q and let v_Q be the corresponding valuation. Let $\phi \in Q$ be a potential key polynomial corresponding to $\Psi(Y)$. Let $\mu = v_Q(\phi)$. Then for each positive integer n consider the inductive commensurable extension V_n of V_Q obtained by setting $v_n(\phi) = \mu + 1/n$. Let U be a nonprincipal ultrafilter on the collection $\{V_n\}_{n\geq 0}$. By Lemma 1.29 we can define the ultrafilter limit domain V_U. First we observe that $V_U \subseteq V_Q$. To see this, let $\rho \in V_U$. Write $\rho = f/g$ and then write

$$f(X) = a_s(X)(\phi(X))^s + \ldots + a_1(X)(\phi(X)) + a_0(X)$$

and:

$$g(X) = b_m(X)(\phi(X))^m + \ldots + b_1(X)(\phi(X)) + b_0(X),$$

with $deg(a_i) < deg(\phi)$ and $deg(b_j) < deg(\phi)$ for each $i = 1, \cdots, s$ and $j = 1, \cdots, m$. Then, when we extend V_Q to V_n we obtain:

$$\begin{aligned} v_n(\rho) &= v_n(f) - v_n(g) \\ &= \min_{i=1,\cdots,s}\{v_Q(a_i) + i(\mu + 1/n)\} \\ &\quad - \min_{j=1,\cdots,m}\{v_Q(b_j) + j(\mu + 1/n)\}. \end{aligned}$$

Since $\rho \in V_U$ we must have $v_n(\rho) \geq 0$ for infinitely many values of n. This means that $1/n$ can be made as small as we wish, which means that the above equation implies that:

$$v_Q(\rho) = \min_{i=0,\cdots,s}\{v_Q(a_i) + i(\mu)\} - \min_{j=0,\cdots,m}\{v_Q(b_j) + j(\mu)\} \geq 0.$$

It is then easy to see that $v_Q(\rho) \geq 0$. Hence $V_U \subseteq V_Q$. Since V_U is a valuation domain there are only two possibilities. Either V_U is one of the upside-down domains corresponding to V_Q or $V_U = V_Q$. Choose positive integers r, t so that $v_Q(\phi^r/p^t) = 0$. Then observe that $v_n(\phi^r/p^t) > 0$ for all n. It follows that ϕ^r/p^t is a nonunit in each V_n and so must be a nonunit in V_U. But it is a unit in V_Q. Hence V_U is a proper subring of V_Q. Moreover, the fact that ϕ^r/p^t is a nonunit indicates that we have shown that V_U is the pullback valuation domain corresponding to the valuation domain $F[Y]_{(\Psi)}$. Hence, $V_U = V$.

(2U) $V/Q = F[Y]_{(Y)}$. The argument here is essentially the same as in case (1U), but with a slightly different set-up. Again, we begin with V_Q being commensurable. Again, let v_Q be the valuation corresponding to V_Q. This time however we view V_Q as being an extension of a domain V_+. Let ϕ be a key polynomial used to accomplish the extension and let $\mu = v_Q(\phi)$. Then define the valuation domain V_n constructed by extending V_+ with the same key polynomial ϕ, but setting $v_n(\phi) = \mu + 1/n$. The proof then proceeds exactly as in case (1U).

(3U) $V/Q = F[1/Y]_{(1/Y)}$. We proceed as in case (2U), except that we set $v_n(\phi) = \mu - 1/n$. Then, instead of arguing using ϕ^r/p^t we make a similar argument using p^t/ϕ^r.

The MacLane extension construction.

(1M) Let V be a commensurable domain and ϕ be a potential key polynomial to be used for extending V. The value group of V is a subgroup of the additive group of the rational numbers. Let v be the valuation associated with V, normalized so that $v(p) = 1$. Let $v(\phi) = \mu$. The method for extending V that we have considered before is to assign a new real number value to ϕ which is larger than μ. Let Γ be the value group of v. We now treat Γ as being the subgroup $\Gamma \oplus 0$ of the direct sum $\mathbb{Q} \oplus \mathbb{Q}$, ordered lexicographically. We then mimic MacLane's method of extending v but rather than assigning a new real value to ϕ we create a new valuation w by assigning $w(\phi) := (v(\phi), 1)$ with the standard lexicographic order. MacLane's proofs ([19, Section 4]) easily demonstrate that w is a valuation. It is, moreover, easy to see that the resulting valuation domain is the upside-down domain obtained by the case (1U) method considered above.

(2M) Again, let V be a commensurable domain with corresponding valuation v. Let V be an extension of a commensurable domain V_+ with corresponding valuation v_+ as in Case (2U), using a key polynomial ϕ. Let $v(\phi) = \mu$. Then we extend V_+ as in case (1M) by viewing the value group of v_+ as being the subgroup $\Gamma \oplus 0$ of the direct sum $\mathbb{Q} \oplus \mathbb{Q}$ and then setting $w(\phi) := (v(\phi), 1)$. Then, again, w is a valuation and yields the upside-down domain of case (2U).

(3M) This case is exactly like case (2M), except that we set $w(\phi) := (v(\phi), -1)$. Then w corresponds to the upside-down domain of case (3U).

What we have accomplished is to show that all of the upside-down domains described using pullbacks can be constructed by means of ultrafilters and by means of extensions with key polynomials - with an extended value group.

In the following we will refer to the upside-down valuation domains of described in (1M) (resp. (2M) and (3M)) as Case 1 (resp. 2 and 3) upside-down valuation domains.

The upside-down valuation domains are different from all of the valuation domains considered thus far in that they are two-dimensional with the prime number p being contained in the height-one prime. Note that the residue field of such a domain is necessarily finite since it is a residue field of $F[Y]_{f(Y)}$ (so, it is algebraic over F, which is finite).

We now turn to the associated polynomial domain. Let V_{usd} be an upside-down valuation domain. As with the previous classes of domains we define $D_{usd} = V_{usd} \cap \mathbb{Q}[X]$ and call D_{usd} an upside-down polynomial domain. And we again designate P_{usd} to be the valuation prime (the contraction of the maximal M ideal of V_{usd} to D_{usd}). Since $D_{usd}/P_{usd} \subseteq V_{usd}/M$, which is finite, the following result is then immediate.

Proposition 1.30. *P_{usd} (cases 1 and 2) is a maximal ideal of D_{usd}.*

Recall that there is a commensurable domain V which is an overring of V_{usd}. It is natural then to wonder about the connection between $D_{usd} = V_{usd} \cap \mathbb{Q}[X]$ and $D_V = V \cap \mathbb{Q}[X]$. In most cases there is a very clear connection. Let Q be the height-one prime of V (so that $V = (V_{usd})_Q$.) Then we have already noted that V/Q is isomorphic to a field $F(Y)$ of rational functions over a finite field F. We are taking Y to be the homomorphic image of a polynomial in $Q[X]$. Then we have the three cases considered above.

Proposition 1.31. *Let D_{usd}, D_V, and Q be as in the preceding paragraph. Suppose that D_{usd} is an upside-down domain considered in either Case 1 or Case 2. Then $D_{usd} = D_V$. In other words, the intersection of the upside-down domain with $\mathbb{Q}[X]$ yields the same result as if we intersected the corresponding inductive valuation overring with $\mathbb{Q}[X]$.*

Proof. First we note that V_{usd} is a subring of V so $D_{usd} \subseteq D_V$ is clear. Suppose then that $f \in D_V$. If $f \in Q$, since Q is also a prime ideal of V_{usd}, then $f \in D_{usd}$. So we suppose that f is a unit in V. Recall that MacLane proved that $D_V/(Q \cap D_V) = F[Y]$, with Y chosen as we have done in Lemma 1.6. Then the image of f in the field $V/Q = F(Y)$ is actually a polynomial in Y. But the pullback domain V_{usd} corresponds to the inverse images of rational functions in a localization of $F[Y]$. So, in particular, it contains the inverse images of all of the polynomials in Y. Hence $f \in D_{usd}$. $\square$

It should be noted that when V is commensurable the polynomial domain D_V is two-dimensional Noetherian, but the maximal ideal of V restricts to a nonmaximal prime ideal. We have just demonstrated that when V is an upside-down domain of either Case 1 or 2, then we still get the two-dimensional Noetherian domain of the commensurable case, but now the maximal ideal of the valuation domain restricts to a maximal, height-two prime ideal.

If V_{usd} is a Case 3 upside-down domain the structure of D_{usd} is somewhat different. In fact, in this case, D_{usd} is very similar to the polynomial domain obtained from an incommensurable domain.

First, let the notation be as above except that V_{usd} is assumed to be a Case 3 upside-down domain. As we did in looking at the ultrafilter construction of upside-down domains, we consider V to be a MacLane extension of a commensurable domain V_+ using the key polynomial ϕ

and the value $v(\phi) = \mu$. This is useful in the present context because ϕ is not a residue unit in V. If we choose positive integers r, t so that $v(\phi^r/p^t) = 0$ then using the canonical homomorphism H from V to $F(Y)$ we have $H(\phi^r/p^t) = Y^m$, for some $m > 0$. This provides the set-up for the following result.

Lemma 1.32. *P_{usd} (Case 3) is a height-two prime ideal of D_{usd}.*

Proof. Consider the polynomial ϕ. We know by construction that ϕ is an irreducible polynomial in $\mathbb{Q}[X]$. Consider the prime ideal $P_\phi = \phi\mathbb{Q}[X] \cap D_{usd}$. Certainly this is a nonzero prime ideal of D_{usd}. We want to prove that $P_\phi \subseteq P_{usd}$. Let v_{usd} be the valuation associated with V_{usd}. Then the value group of v_{usd} is a direct sum of the form $\Gamma \oplus \mathbb{Z}$, ordered lexicographically, where Γ is an additive subgroup of the rational numbers. We can normalize so that $v_{usd}(p) = (1, 0)$. Since the maximal ideal of V_{usd}/Q is generated by $1/Y$ and $H((\phi)^r/p^t) = Y^m$, we conclude that $v_{usd}(\phi) = (a, b)$ where $a > 0$ and $b < 0$. Let $(\phi f)/p^t \in P_\phi \backslash P_{usd}$, with $f \in \mathbb{Q}[X]$ and $v_{usd}(f) = (c, d)$. Then $v_{usd}(\phi f/p^t) = 0$ and $v_{usd}(p^t) = (t, 0)$. All of this forces us to conclude that $d > 0$. This is inconsistent with our knowledge that $V_{usd}/Q = F[1/Y]_{(1/Y)}$ and $D_V/Q = F[Y]$. Then f would have to simultaneously correspond to a polynomial in Y and $1/Y$ which was not a constant. This proves that $P_\phi \subseteq P_{usd}$. Since we also know that $p \in P_{usd}$ and $p \notin P_\phi$, this proves that P_{usd} must have height two. $\square$

Lemma 1.3 indicates that the radical of the principal ideal generated by p is a prime ideal of D_{usd}. In fact, it is easy to see from the proof of the preceding lemma that the prime ideal P_ϕ is in the radical of (p). Since p is also in the radical of (p) and $p \notin P_\phi$ then P_{usd} must be the radical of (p). Just as in the case of the inductive incommensurable domain, this proves that P_{usd} is the unique height-two prime of D_{usd}.

We collect together some immediate results that follow from our investigation so far.

Corollary 1.33. *Let V_{usd} be a Case 3 upside-down domain as above. Assume all the above notation as well. Then*

(1) *P_{usd} is the unique height-two prime ideal of D_{usd}.*

(2) *P_{usd} is the only prime ideal of D_{usd} which contains p.*

(3) *D_{usd} is not Noetherian.*

(4) *The residue field D_{usd}/P_{usd} is finite.*

Thus far we have been considering valuation domains and domains of the form $V \cap \mathbb{Q}[X]$ where V is a valuation domain. In this last case where V is a Case 3 upside-down domain, it is worth noting that we have another representation of D_{usd} as an intersection of $\mathbb{Q}[X]$ with a quasi-local domain.

We recall that a domain D with quotient field K is a *pseudo-valuation domain* if every prime ideal P of D is *strongly prime*, that is if $x, y \in K$, $xy \in P$ and $x \notin P$, then $y \in P$. By [2] D is pseudo-valuation if and only if it is a pullback of a valuation domain V, as follows:

$$\begin{tikzcd} D \arrow[r] \arrow[d] & k \simeq D/\mathfrak{m} \arrow[d] \\ V \arrow[r] & L \simeq V/\mathfrak{m} \end{tikzcd}$$

where $\mathfrak{m}$ is the maximal ideal of V, k is a subfield of L, the horizontal arrows are the natural projections and the vertical arrow are injections.

Theorem 1.34. *Let V_{usd} be a Case 3 upside-down valuation domain and V_Q be its inductive valuation overring (i.e. V_Q is the localization of V_{usd} at the height one prime Q). Then $D_{usd} = V_{usd} \cap \mathbb{Q}[X] = A \cap \mathbb{Q}[X]$, where A is a suitable pseudo-valuation domain properly contained in V_{usd}.* ◻

Proof. Recall that $V_{usd}/Q = F[1/Y]_{(g(1/Y))}$. It is clear that $V_{usd} \cap \mathbb{Q}[X] \subset V_Q \cap \mathbb{Q}[X]$ since $V_{usd} \subseteq V_Q$. Let $\{V_i\}_{i \in I}$ be the family of all Case 1 or 2 upside-down valuation domains which are pullbacks of $V_Q = W$, and consider $A := V_{usd} \cap (\bigcap_{i \in I} V_i)$. Then it is easy to check that A is a pullback of the following type:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & F \arrow[d] \\ W \arrow[r] & F(Y) \end{tikzcd}$$

whence A is a pseudo-valuation domain. Recall that $V_i \cap \mathbb{Q}[X] = V_Q \cap \mathbb{Q}[X]$. The result follows as is shown below:

$$\begin{aligned} A \cap \mathbb{Q}[X] &= V_{usd} \cap (\bigcap_{i \in I, V_i \neq V_{usd}} V_i \cap \mathbb{Q}[X]) \\ &= V_{usd} \cap V_Q \cap \mathbb{Q}[X] = V_{usd} \cap \mathbb{Q}[X]. \end{aligned}$$

□

Remark 1.35. We point out that the p-unitary valuation overrings of $\mathrm{Int}(\mathbb{Z}_p)$ are never upside-down. This is a general result. In fact, it can easily be deduced from what we have already done that if D is a Prüfer domain such that $\mathbb{Z}_p[X] \subseteq D \subseteq \mathbb{Q}[X]$, then its p-unitary valuation overrings are all limit (Theorem 4.2).

2. Ordering the valuation domains. In this section we make explicit the ordering of the valuation domains that has been alluded to earlier. We also indicate how the ordering will be used in analyzing the prime ideal structure of integrally closed domains between $\mathbb{Z}_p[X]$ and $\mathbb{Q}[X]$.

Definition 2.1. Let V and W be two valuation domains in T_p with maximal ideals M and N, respectively. We write $V \preceq_{Mac} W$ provided:

(1) $V \cap \mathbb{Q}[X] \subseteq W \cap \mathbb{Q}[X]$ $\quad (D_V \subseteq D_W)$ and

(2) $M \cap \mathbb{Q}[X] \subseteq N \cap \mathbb{Q}[X]$ $(M \cap D_V \subseteq N \cap D_W)$.

It is useful, at this point, to recall what has been stated in Section 1 with regards to the radical of p in the domains of the type D_V, with $V \in T_p$.

Lemma 2.2. *Let $V \in T_p$, with maximal ideal M, and consider the domain $D_V = V \cap \mathbb{Q}[X]$. The radical of p in D_V is the valuation prime $M \cap \mathbb{Q}[X]$ if and only if V is an inductive (commensurable or incommensurable), limit or Case* 3 *upside-down valuation domain. If V is a Case* 1 *or* 2 *upside-down valuation domain, then the valuation prime is a height-two prime strictly containing the radical of p (which is, instead, the valuation prime of the commensurable domain associated with V, and is height-one).*

Lemma 2.3. *Let V and W be as in Definition* 2.1 *and suppose that V is not a Case* 1 *or* 2 *upside-down valuation domain. Then $D_V \subseteq D_W$ if and only if $V \preceq_{Mac} W$.*

Proof. Suppose that $D_V \subseteq D_W$. Since V is not a Case 1 or 2 upside-down valuation domain, the valuation prime $M \cap D_V$ is the radical of p (Lemma 2.2), whence $M \cap D_V \subseteq N \cap D_V \subseteq N \cap \mathbb{Q}[X]$. Thus, $V \preceq_{Mac} W$.

The converse directly follows from Definition 2.1. □

We observe that if V is a Case 1 or 2 upside-down domain, associated with a commensurable domain V', then $D_V = D_{V'}$ but $M \cap \mathbb{Q}[X] \supset M' \cap \mathbb{Q}[X]$ (M and M' are, respectively, the maximal ideals of V and V'). Thus $V \succ_{Mac} V'$.

Lemma 2.4. *Suppose we are given two MacLane valuation domains $V, W \in T_p$, with V inductive commensurable. If $D_V \subseteq D_W$, then $v(f) \leq w(f)$, for each $f \in \mathbb{Q}[X]$.*

Proof. Let v be the valuation associated with V normalized so that $v(p) = 1$. Since each polynomial in $\mathbb{Q}[X]$ can be written in the form f/p^t, with $f \in \mathbb{Z}_p[X]$, it is sufficient to prove the desired inequality for polynomials in $\mathbb{Z}_p[X]$. Look at all the fractions of the type f^n/p^t. Suppose that t_0/n_0 is the maximum (or supremum) of the fractions t/n such that $f^n/p^t \in D_V$. This means that f^{n_0}/p^{t_0} is a unit in V. If not, $v(f^{n_0}/p^{t_0}) > 0$, whence $v(f) > t_0/n_0$, and taking t' and n' such that $v(f) > t'/n' > t_0/n_0$, we have $v(f^{n'}/p^{t'}) > 0$ and so $f^{n'}/p^{t'} \in D_V$. Thus, $v(f^{n_0}/p^{t_0}) = 0$ and $v(f) = t_0/n_0$. Obviously, since $D_V \subseteq D_W$, if

$f^n/p^t \in D_V$ then $f^n/p^t \in D_W$. So the maximum (or supremum) of the fractions t/n such that $f^n/p^t \in D_V$ is less or equal than the analogue for D_W. Then $v(f) \leq w(f)$. □

Proposition 2.5. *Suppose we are given two MacLane valuation domains $V, W \in T_p$, with V inductive commensurable. Then, $V \preceq_{Mac} W$ if and only if W is an extension of V.*

Proof. If W is an extension of V, then $D_V \subseteq D_W$ by the monotonic property of MacLane's extensions (see page 12). By Lemma 2.3, $V \preceq_{Mac} W$.

Conversely, suppose that $V \preceq_{Mac} W$. Then $D_V \subseteq D_W$ and $M \cap \mathbb{Q}[X] \subseteq N \cap \mathbb{Q}[X]$, where M and N are respectively the maximal ideals of V and W. Set $M_w := N \cap D_V$, which is a prime ideal containing p. If M_w is height-one, then $(D_V)_{M_w} = (D_V)_{M \cap \mathbb{Q}[X]} = V \subseteq W$ (Lemma 1.3), whence $V = W$.

If M_w is height-two, we want to apply the same argument of [19, Theorem 8.1]. In order to do this, we have to show that starting from $V = V_0$, it is possible to construct a sequence of inductive valuation domains $\{V_k\}_{k \geq 0}$, such that, for each polynomial f the following conditions are verified:

(I) $w(f) \geq v_k(f)$;

(II) $\deg(f) < \deg(\phi_k)$ implies that $w(f) = v_k(f)$;

(III) $w(\phi_k) = v_k(\phi_k)$,

where ϕ_k is the key-polynomial used to construct v_k as an extension of v_{k-1}.

<u>Construction of V_1.</u> Choose a key polynomial ϕ_1 corresponding to M_w and set $\rho := w(\phi_1)$. Consider V_1, the first stage extension of V obtained by using the key polynomial ϕ_1 with value ρ. Then, $V_1 \preceq_{Mac} W$. In fact, take any $f \in \mathbb{Q}[X]$ and write it as

$$f = a_n\phi^n + a_{n-1}\phi^{n-1} + \ldots + a_1\phi + a_0,$$

with $\deg(a_i) < \deg(\phi)$, for $i = 0, \cdots, n$. Then:

$$\begin{aligned} w(f) &\geq \min_{i=1,\cdots,n}\{w(a_i)+iw(\phi)\} \\ &\geq \min_{i=1,\cdots,n}\{v(a_i)+i\rho\} \text{ (Lemma2.4)} \\ &= v_1(f). \end{aligned}$$

This shows first that $V_1 \cap \mathbb{Q}[X] \subseteq W \cap \mathbb{Q}[X]$ and, by Lemma 2.3, $V_1 \preceq_{Mac} W$. Moreover property (I) is verified for v_1. Properties (II) and (III) follow from the construction of v_1.

Construction of V_k. Applying the above procedure to W and V_{k-1} we can construct an inductive valuation domain V_k still verifying properties (I), (II) and (III) (with $V_1 \preceq_{Mac} V_{k-1} \preceq_{Mac} V_k \preceq_{Mac} W$). Thus we have the desired sequence $\{V_k\}_{k\geq 0}$ and we can state that W is an extension for V. □

Corollary 2.6. *Suppose we are given two valuation domains in T_p, V and W, such that $V \preceq_{Mac} W$ and $V \succeq_{Mac} W$. Then $V = W$.* □

Thus the relation $\preceq_{Mac}$ of Definition 2.1 is an order relation on the set of MacLane valuation domains. In particular, Proposition 2.5 shows that the inductive extension process indicated by MacLane is equivalent to this order and Lemma 2.10 (1) characterizes the limit valuations as the maximal points.

It will be important to note the interaction of the ordering with ultrafilter limits.

Lemma 2.7. *Let $T_\Lambda = \{V_\lambda\}_{\lambda\in\Lambda}$ be an infinite collection of valuation domains in T_p. Suppose that T_Λ is totally ordered (i.e. is a chain) under the order $\preceq_{Mac}$. Suppose also that Λ is given the corresponding ordering (which we simply identify as $\geq$). Then:*

(1) *suppose that the chain is descending. Let U be a nonprincipal ultrafilter on Λ such that each set $B_{\lambda_n} = \{\lambda \in \Lambda | \lambda \leq \lambda_n\}$ is in the ultrafilter. Construct the ultrafilter limit V_U. Then $V_U \preceq_{Mac} V_\lambda$ for each $\lambda \in \Lambda$;*

(2) *suppose that the chain is ascending. Let U be a nonprincipal ultrafilter on Λ such that each set $B_{\lambda_n} = \{\lambda \in \Lambda | \lambda \geq \lambda_n\}$ is in the*

ultrafilter. Construct the ultrafilter limit V_U. Then $V_U \succeq_{Mac} V_\lambda$ for each $\lambda \in \Lambda$.

Proof. (1) Choose a valuation domain V_{λ_n} in the collection. Let $f \in V_U \cap \mathbb{Q}[X]$. We need to show that $f \in V_{\lambda_n} \cap \mathbb{Q}[X]$. We know from the definition of V_U that the set $C(f) = \{\lambda \in \Lambda | f \in V_\lambda\}$ is in U. We know by hypothesis that the set $B_{\lambda_n} = \{\lambda \in \Lambda | \lambda \leq \lambda_n\}$ is also in the ultrafilter. Hence the set $B_{\lambda_n} \cap C(F)$ is in U. This intersection must be nonempty. Choose $\lambda \in B_{\lambda_n} \cap C(F)$. Then we have $f \in V_\lambda$ and $V_\lambda \preceq_{Mac} V_{\lambda_n}$. Hence $f \in V_{\lambda_n} \cap \mathbb{Q}[X]$. It follows that $V_U \preceq_{Mac} V_{\lambda_n}$.

(2) Choose a valuation domain V_{λ_n} in the collection. Let $f \in V_{\lambda_n} \cap \mathbb{Q}[X]$. We need to show that $f \in V_U \cap \mathbb{Q}[X]$. Let $\lambda \geq \lambda_n$. We know from the ordering hypothesis that $f \in V_\lambda \cap \mathbb{Q}[X]$. It follows that $C(f) \subseteq B_{\lambda_n}$ and hence $C(f) \in U$. Hence, $f \in V_U \cap \mathbb{Q}[X]$ and so $V_U \succeq_{Mac} V_\lambda$ for each $\lambda \in \Lambda$. □

The importance of this result (especially (1)) is that it allows us to give an easy Zorn's lemma argument for the following Theorem.

Theorem 2.8. *Let D be an integrally closed domain lying between $\mathbb{Z}_p[X]$ and $\mathbb{Q}[X]$. Then every p-unitary valuation overring of D is comparable to a minimal element under the ordering $\preceq_{Mac}$.*

Notation 2.9. Let D be a domain lying between $\mathbb{Z}_p[X]$ and $\mathbb{Q}[X]$. We denote by T_D the set of valuation domains in T_p which are minimal with respect to being overrings of D under $\preceq_{Mac}$.

Several observations regarding this ordering are true.

Lemma 2.10. (1) *If V is a limit valuation domain, then V is maximal under the ordering $\preceq_{Mac}$.*

(2) *If V is an inductive commensurable domain and V_{usd} is a corresponding Case 1 or 2 upside-down domain, then $V_{usd} \succeq_{Mac} V$.*

(3) *If V is an inductive commensurable domain and V_{usd} is the corresponding Case 3 upside-down domain, then $V \succeq_{Mac} V_{usd}$.*

Proof. (1) Let V be a limit valuation domain with maximal ideal M. Since we obtain V again when we localize $V \cap \mathbb{Q}[X]$ at $M \cap \mathbb{Q}[X]$ we know that no other valuation domain in T_p can have a maximal ideal which contracts to $M \cap \mathbb{Q}[X]$. We also know that $M \cap \mathbb{Q}[X]$ is maximal in $V \cap \mathbb{Q}[X]$ and is the only prime ideal of $V \cap \mathbb{Q}[X]$ which contains p. The result follows easily.

(2) Let V be a commensurable domain and let V_{usd} be a corresponding Case 1 or 2 upside-down domain. Then, by Proposition 1.30, $V_{usd} \cap \mathbb{Q}[X] = V \cap \mathbb{Q}[X]$.

(3) Let V be a commensurable domain and let V_{usd} be a corresponding Case 3 upside-down domain. The result follows immediately from considering the Case 3M construction. □

Lemma 2.11. *Let D be an integrally closed domain lying between $\mathbb{Z}_p[X]$ and $\mathbb{Q}[X]$. Let $V \in T_D$. Then V is not a Case 1 or Case 2 upside-down domain.*

Proof. Suppose that $V \in T_D$ is a Case 1 or Case 2 upside-down domain. Then $V \cap \mathbb{Q}[X]$ is also an overring of D. Let V^* be the commensurable domain corresponding to V. Then $V \cap \mathbb{Q}[X] = V^* \cap \mathbb{Q}[X]$ and so V^* is also an overring of D. Since V^* is strictly less than V under $\preceq_{Mac}$ we have contradicted the minimality of V. □

More can be said about the ordering and the relationship between a commensurable domain and an associated upside-down domain.

Lemma 2.12. *Let V be an inductive commensurable domain. Suppose that W is a valuation domain in T_p which is distinct from V and is greater than V. Then $W \succeq_{Mac} V_{usd}$ for exactly one Case 1 or Case 2 upside-down domain associated with V.*

Proof. If $W \succeq_{Mac} V$, then the maximal ideal of W must lie over a nonzero prime of D_V. Then note that if W is distinct from V, the maximal ideal of W must lie over a height-two prime of D_V, say M. If V_{usd} is a Case 1 or 2 upside-down domain associated with V, then V_{usd} is centered in a height-two maximal ideal of D_V, say P. Now,

$W \succeq_{Mac} V_{usd}$ if and only if $M = P$. Given P, there exists exactly one upside-down domain associated with V which is centered on P. In fact, P corresponds to an irreducible polynomial $f \in F[Y]$, being $F[Y]$ the polynomial ring of V and V_{usd} is the following pullback:

$$\begin{tikzcd} V_{usd} \arrow[r] \arrow[d] & F[Y]_{(f)} \arrow[d] \\ V \arrow[r] & F(Y) \end{tikzcd}$$

The result follows. □

Lemma 2.13. *Let V be an inductive commensurable domain and let W be the corresponding Case 3 upside-down domain. Then any valuation domain V^* which is distinct from W and is greater than W, is also greater than V. In other words, V is the unique minimal extension of W.*

Proof. Let V^* be a valuation domain in T_p which is greater than W and is distinct from W. Let v^*, v, and w be valuations associated with V^*, V, and W respectively. Write the value group of V^* as a direct product $\Gamma_1^* \oplus \Gamma_2^*$ with Γ_1^* and Γ_2^* both being subgroups of the additive group of the rational numbers. If V^* is itself an upside-down domain then we assign Γ_1^* and Γ_2^* as in the constructions Cases $1M, 2M, 3M$ normalized so that $v^*(p) = (1, 0)$. If V^* is not upside-down we assume that $\Gamma_2^* = 0$ and, as before, $v^*(p) = (1, 0)$. (If V^* is an infinite limit domain we assume that $\infty \in \Gamma_1^*$. It is plain from the pullback construction of the upside-down domains that the Case 3 upside-down domain associated with a given commensurable domain is unique. Moreover, any Case 1 or 2 upside-down domain associated with V is clearly greater than W (under $\preceq_{Mac}$). Finally, if V^* is not an upside-down domain associated with V, then in order to have $V^* \succeq_{Mac} W$ it must be that whenever $f \in \mathbb{Q}[X]\backslash\{0\}$ we have $v^*(f) = (a, b)$ and $w(f) = (c, d)$ with $a \geq c$. Then, the uniqueness of the Case 3 upside-down domain associated with V and the fact that $v(f) = c$ implies that $W \succeq_{Mac} V$. □

Suppose V is an inductive commensurable domain. In the MacLane extension method we choose a key polynomial corresponding to one of the height-two primes of the polynomial domain D_V (other than the maximal ideal corresponding to $\Psi(Y) = Y$ - in this case we consider the key polynomial used to construct V). In any case, we have a key polynomial and we extend V to a larger (under $\preceq_{Mac}$) valuation domain W by increasing the value of ϕ. If W is also a commensurable domain then we can choose a key polynomial (avoiding the $\Psi(Y) = Y$ case now) and extend W to an even larger valuation domain. In this larger domain the value of ϕ will not change and ϕ will then become a residue unit. We observe next that this phenomenon in which the value of ϕ remains unchanged in subsequent extensions is unique to the situation where W is a commensurable domain.

Proposition 2.14. *Let V be an inductive commensurable domain with associated valuation v and let ϕ be either a potential key polynomial to be used for extending V or else a key polynomial used to extend a smaller domain to obtain V. Let W be a domain in T_p with associated valuation w and suppose that $W \succeq_{Mac} V$ and that $w(\phi) = r > v(\phi)$ for some irrational number r. Then W is the simple first stage extension obtained from V by assigning the value $w(\phi) = r$. In other words, there is only one valuation domain W greater than V such that $w(\phi) = r$.*

Proof. First observe that W cannot be an upside-down domain or a limit domain because the value groups of such domains are either contained in the rational numbers or are the direct sum of two groups, each of which is contained in the rational numbers. So W must be an incommensurable valuation domain.

Suppose that ϕ is a proper key polynomial over V, that is $\phi(Y) \neq Y$ (which means, as we explained at page 9, that ϕ has not been used to construct V from a previous inductive valuation domain). Recall that we can write a unique homogeneous chain of inductive valuation domains starting from $\mathbb{Z}_p$ whose last member is W (Lemma 1.28) and we can do the same with V. By Proposition 2.5, W is an extension of V in the sense of Definition 1.27. By ([19, Theorem 16.3]), each inductive or limit valuation domain is homogeneous. Thus we have a homogeneous chain from $\mathbb{Z}_p$ to W, a homogeneous chain from $\mathbb{Z}_p$ to V and a homogeneous chain from V to W. The unicity of these chains implies that V is a member of the chain going from $\mathbb{Z}_p$ to W. Suppose ϕ is the key polynomial used to extend to the next member after V. If the successor to V is also commensurable, then the value of ϕ either remains constant at a rational number or is extended to a larger rational number at which it stays for the rest of the sequence. This contradicts our assumption that $w(\phi) = r$ is irrational. If the successor to V in the homogeneous chain is incommensurable, then the successor to V is W because only the last member of the chain can be incommensurable. This proves the result in the case where ϕ is a proper key polynomial.

Suppose that V is an extension of a commensurable domain V_1 with the extension accomplished using the key polynomial ϕ. Then V_1 is a member of the homogeneous chain representing W and the argument goes essentially the same as that above. The value of ϕ must increase and it must increase to a value larger than $v(\phi)$. If it increases to a rational number smaller than r we reach a contradiction, and if it increases to r in the successor to V_1 then W is a simple extension of V_1 using the key polynomial ϕ. This proves the result. □

A similar result is true for extensions of V in which the value of ϕ resides in a rank two group.

Proposition 2.15. *Let V be an inductive commensurable domain with associated valuation v and let ϕ be either a potential key polynomial to be used for extending V or else a key polynomial used to extend a smaller domain to obtain V. Let W be a domain in T_p with associated valuation w and suppose that $W \succeq_{Mac} V$ and that $w(\phi) = (t, r)$ with $t \geq v(\phi)$ for some rational numbers t, r. (We assume here that $w(p) = (1, 0)$). Then W is the simple extension obtained from V by assigning the value $w(\phi) = (t, r)$. In other words, there is only one valuation domain W greater than V such that $w(\phi) = (t, r)$.*

(Note that there are two possible cases here. If $r > 0$, then W would be a Case 1 or 2 upside-down domain, while $r < 0$ would imply that W is a Case 3 upside-down domain.)

Proof. It is clear that W must be an upside-down domain. W must then be associated with a commensurable domain W^* such that $w^*(\phi) = (t, 0)$. One possibility then would be that W^* is the simple extension of V obtained by assigning the value $w^*(\phi) = (t, 0)$. This would finish the result. Suppose this is not the case, so $r \neq 0$. Then, as in the preceding result we express W^* as the last term in a homogeneous sequence of commensurable domains. Then, as before, we consider the domain in the homogeneous chain which is the smallest term which is still larger than V.

• If the value of ϕ does not increase in this extension over V then ϕ becomes a residue unit. This is a contradiction since the pullback construction of the upside-down domain indicates that the polynomials which have value (t, r) with $r \neq 0$ are exactly those which correspond to nonconstant polynomials in the residue ring $F[Y]$, i.e. those which are not residue units.

• If the value of ϕ does increase in this extension then the immediate successor to V must be the last term in the sequence, i.e. it must be W^*. The reason for this is that if not, then in the next extension ϕ becomes a residue unit and we face the same problem as before. Hence, W^* must be a simple extension of V obtained using ϕ. Then the associated upside-down domain W is also a simple extension of V.

This finishes the result. □

Remark 2.16. When the value (t, r) is assigned in the above proof, the only concern regarding r is whether it is positive or negative. For any chosen value of t there are exactly two corresponding upside-down domains depending on whether $r > 0$ (either case 1 or 2) or $r < 0$ (Case 3).

We now turn back to the question of the minimal elements under the ordering $\preceq_{Mac}$. Let V be a commensurable domain and let D be a domain. The question we address is how the valuation overrings of D relate under $\preceq_{Mac}$ to V. Recall that (Lemma 2.12) if $W \in T_p$ and $W \succeq_{Mac} V$ then either $W = V$ or $W \succeq_{Mac} V_{usd}$ for exactly one Case 1 or 2 upside-down domain V_{usd} associated with V. So we turn now to restricting the collection of valuation domains which can be minimal over D with respect to $\preceq_{Mac}$ and greater than V by comparing them to the upside-down domains.

Proposition 2.17. *Assume that D is an integrally closed domain lying between $\mathbb{Z}_p[X]$ and $\mathbb{Q}[X]$. Let $V \in T_p$ be an inductive commensurable domain. Let $S = \{V_i | i \in I\}$ be a collection of valuation domains in T_p which are valuation overrings of D, minimal with respect to being overrings of D under $\preceq_{Mac}$. Suppose also that $V_i \succeq_{Mac} V$ for each $i \in I$. Then there are only a finite number of upside-down domains V_{usd} associated with V such that $V_i \succeq_{Mac} V_{usd}$, for some $i \in I$.*

Proof. Suppose not. There is only one Case 2 upside-down domain and one Case 3 domain associated with V. So we let J be an infinite subset of I and $\{W_j | j \in J\}$ be an infinite collection of Case 1 upside-down domains associated with V, such that for each $j \in J$ there exists a unique domain $V_j \in S$ with $V_j \succeq_{Mac} W_j$. Let U be a nonprincipal ultrafilter on J and use it to define the ultrafilter limit domain V_U of the collection $\{V_j | j \in J \subseteq I\}$. Since $V_j \succeq_{Mac} V$ for each $j \in J$, it is easy to see that $V_U \succeq_{Mac} V$. We claim that $V_U = V$. We know from Lemma 2.12 that if $V_U \neq V$ then there must be a Case 1 or Case 2 upside-down domain V_{usd} associated with V such that $V_U \succeq_{Mac} V_{usd}$. Hence, the maximal ideal of V_U lies over a height-two maximal ideal of D_V. Choose an element f of this maximal ideal which is a unit in V (i.e. not contained in the height-one valuation prime). Note that each V_j is greater than a distinct upside-down domain, so we can similarly deduce that the maximal ideals of the collection $\{V_j | j \in J \subseteq I\}$ of valuation domains each lie over distinct height-two maximal ideals of D_V. Since f is a nonunit in V_U it must be a nonunit in an infinite number of the valuation domains in the collection $\{V_j | j \in J \subseteq I\}$ used to construct V_U. Let P be the contraction to D_V of the maximal ideal of V. Then D_V/P is isomorphic to a ring $F[Y]$ of polynomials and the element f must correspond to an element of $F[Y]$ which lies in an infinite number of maximal ideals. Hence, f must map to 0 in $F[Y]$ which means that f is in P. But we chose f to not be in P. This contradiction proves that $V_U = V$. This finishes the proof because each V_j is an overring of D which implies that V_U is also an overring of D. This contradicts the minimality of the $V_j's$. □

Remark 2.18. Proposition 2.17 gives a strong finiteness condition on the set of minimal elements of the p-unitary valuation overrings of an integrally closed domain D lying between $\mathbb{Z}_p[X]$ and $\mathbb{Q}[X]$. In particular, if V is a commensurable valuation domain and we want to build a minimal set of p-unitary valuation overrings of D which are greater than V using MacLane extensions, then we need only consider key polynomials corresponding to finitely many maximal ideals of D_V when we are extending V. We now fix a commensurable domain V and an integrally closed domain D as above and also fix a maximal ideal M of D_V. The question we investigate now is what can be said about the valuation overrings of D which are greater than V and are centered on M.

Proposition 2.19. *Let D, V and M be as above. Let $S_{V,M} = \{V_i | i \in I\}$ be the family of valuation overrings of D, minimal with respect to being overrings of D under $\preceq_{Mac}$, greater than V and centered on M. For each $i \in I$, let $W_i = V_i$ if V_i is limit or inductive; and if V_i is upside-down, let W_i be the inductive commensurable domain corresponding to V_i. Let ϕ be a potential key polynomial for V corresponding to M and let w_i be the valuation corresponding to W_i. Let $\rho = \inf_{i \in I}\{w_i(\phi)\}$. Then there is a valuation domain V_i in $S_{V,M}$ such that $w_i(\phi) = \rho$.*

Proof. If $S_{V,M}$ is finite the result is trivial. So we assume it is infinite. Suppose then that there is no such V_i. Let U be a nonprincipal ultrafilter on I. Suppose that for each real number $\tau > \rho$ the set $\{i \in I | w_i(\phi) < \tau\}$ is in U. Define the ultrafilter limit valuation domain V_U. Then V_U is also an overring of D which is centered on M. If the value group of V_U is contained in the real numbers (plus ∞) then $v_U(\phi) = \rho$ is evident. This is a contradiction because V_U would not be in $S_{V,M}$ since $v_U(\phi)$ is too small and yet it cannot be larger than any of the minimal elements contained in $S_{V,M}$. Suppose then that V_U is an upside-down domain. Then it follows easily that the corresponding commensurable domain W_U satisfies $w_U(\phi) = \rho$. As before, this yields a contradiction. $\square$

Theorem 2.20. *Assume the notation of the previous two results. One of the following is true.*

(1) *Every valuation domain in $S_{V,M}$ is an infinite limit extension with $w(\phi) = \infty$.*

(2) *$S_{V,M}$ contains exactly one valuation domain, which we designate as W. This domain W must be one of the following types:*

(a) *an inductive commensurable extension of* V *obtained by setting* $w(\phi) = \rho$

(b) *an inductive incommensurable extension of* V *obtained by setting* $w(\phi) = \rho$ *with* ρ *an irrational number.*

(c) *a Case 3 upside-down domain associated with the inductive commensurable domain of (a).*

(3) *The value* ρ *is a rational number and the extension* W *of* V *obtained by setting* $w(\phi) = \rho$ *has the following properties.*

(a) $W \notin S_{V,M}$.

(b) *Every domain in* $S_{V,M}$ *is strictly greater than* W.

(c) *In at least one of the domains in* $S_{V,M}$ *the polynomial* ϕ *has value* ρ.

Proof. It is clear that (1) occurs if and only if $\rho = \infty$. Suppose then that ρ is a real number.

Consider the simple inductive extension (V, ρ) of V obtained by assigning ρ as the value of ϕ. Let V_i be a domain in $S_{V,M}$ such that $v_i(\phi)$ is real. By hypothesis $v_i(\phi) \geq \rho$. It follows easily then that $V_i \succeq_{Mac} (V, \rho)$. Let $V_i \in S_{V,M}$ such that $v_i(\phi)$ is not real. Then either $\rho = \infty$ (which we have already dealt with) or V_i is an upside-down valuation domain. Lemma 2.12 implies that a Case 1 or 2 upside-down valuation domain cannot be in $S_{V,M}$. Hence, we are left with the case where V_i is a Case 3 upside-down valuation. Suppose that $v_i(\phi) = (r, t)$ with $t < 0$. If $r > \rho$, then $V_i \succeq_{Mac} (V, \rho)$. Suppose that $r = \rho$. Then Proposition 2.15 implies that V_i is the Case 3 upside-down domain associated with (V, ρ). In this case (V, ρ) is greater than V_i and it follows that $S_{V,M}$ consists of just V_i. So, given that ρ is real, we have two possibilities:

(1) $S_{V,M}$ consists of only the Case 3 upside-down valuation domain associated with (V, ρ).

(2) Every domain in $S_{V,M}$ is greater than (V, ρ).

This second case above has several subcases.

(a) Suppose that ρ is irrational. Proposition 2.14 implies that there is only one valuation domain greater than V in which ϕ has value ρ, that domain being (V, ρ). Since an incommensurable domain does not admit any associated upside-down domain, in any valuation domain which is greater than (V, ρ) the polynomial ϕ must have value greater than ρ. Proposition 2.17 then implies that (V, ρ) itself is in $S_{V,M}$. In this case also then, we have only one domain in $S_{V,M}$.

(b) Suppose that ρ is rational and that (V, ρ) is in $S_{V,M}$. In this case (V, ρ) is the only domain in $S_{V,M}$.

(c) Suppose that ρ is rational and that (V, ρ) is not in $S_{V,M}$. We know from Proposition 2.17 that ϕ must have value ρ either in some member of $S_{V,M}$ or in the commensurable domain associated with some Case 3 upside-down domain in $S_{V,M}$. The Case 3 upside-down domain associated with V, ρ is not in $S_{V,M}$ (since it is less than (V, ρ)). It follows that there must be a domain W in $S_{V,M}$ in which some polynomial has greater value than in (V, ρ) and yet ϕ has value ρ. In other words W is greater than some proper MacLane extension of (V, ρ) using a key polynomial associated with a different maximal ideal of $(V, \rho) \cap \mathbb{Q}[X]$ than ϕ. $\square$

3. A graphic representation of MacLane ordering. MacLane's paper starts with the p-adic valuation on the field of rational numbers and gives an iterative process which, if continued indefinitely, can be used to build any p-unitary valuation domain with quotient field $\mathbb{Q}(X)$. Moreover, the process places a natural partial ordering on the resulting valuation domains. In the previous section we took an arbitrary domain D lying between $\mathbb{Z}_p[X]$ and $\mathbb{Q}[X]$ and analyzed the collection $T_D = \{W_i\}_{i \in I}$ of valuation overrings of D which are minimal under the MacLane ordering. In this section we start with the p-adic valuation on $\mathbb{Q}$ and build toward the valuation domains in T_D. This process can be naturally thought of as the building of a (perhaps infinite) tree. The branches of the tree correspond precisely to the valuation domains in T_D. The vertices which are not terminal points of a branch are not in T_D but are stepping stones in the building process. We will denote this tree with $\mathcal{T}_D$.

The key feature of this tree is that at each stage of building we extend from a given vertex to only a finite number of adjacent vertices. We refer to this feature by saying that the tree is locally finite. We approach this from two different directions.

(1) We start with a domain D and show that we can represent the corresponding family of valuation domains T_D by means of a locally finite tree.

(2) We consider a locally finite tree and show that there is a corresponding domain D such that this tree represents the collection T_D.

Construction of the tree $\mathcal{T}_D$. First we set/recall some notation. Let D be a domain, let V be a p-unitary valuation domain, and let M be the center of the maximal ideal of V in the domain $D_V = V \cap \mathbb{Q}[X]$. Finally, let $S_{V,M}$ be the collection of p-unitary valuation domains which are

- overrings of D;
- minimal amongst valuation overrings of D with respect to $\preceq_{Mac}$;
- centered on M;
- greater than V under $\preceq_{Mac}$.

As above, we let T_D represent the collection of all p-unitary valuation overrings of D which are minimal under $\preceq_{Mac}$. It is clear then that for any choice of V and M we have $S_{V,M} \subseteq T_D$.

Note that under $\preceq_{Mac}$ every p-unitary valuation domain is greater than $\mathbb{Z}_p(X)$. Consider then the polynomial domain $\mathbb{Z}_p(X) \cap \mathbb{Q}[X] = \mathbb{Z}_p[X]$. One of two things is true then:

(1) $T_D = \{\mathbb{Z}_p(X)\}$ and $D = \mathbb{Z}_p[X]$.

or

(2) Each V in T_D is centered on a maximal ideal M of $\mathbb{Z}[X]$.

Suppose the latter possibility above is true. Then Remark 2.18 implies that only a finite number of maximal ideals can occur. Denote these maximal ideals by $M_1, M_2, \ldots, M_n$. Then we can partition T_D into the sets $S_{\mathbb{Z}_p, M_i}$.

For each i choose a potential key polynomial $\phi_i(X)$. Each domain in $S_{\mathbb{Z}_p, M_i}$ must be either an inductive domain, a limit domain or a Case 3 upside-down domain. As in Proposition 2.19, for each V_λ in $S_{\mathbb{Z}_p, M_i}$ we let $W_\lambda = V_\lambda$ if V_λ is inductive or limit; and if V_λ is Case 3 inductive then let W_λ be the inductive domain corresponding to V_λ. Then let

w_λ be the valuation corresponding to W_λ. As in Proposition 2.19, let $\rho_i := \inf\{w_\lambda(\phi_i) \mid W_\lambda \in S_{\mathbb{Z}_p,M_i}\}$.

Now for each i we want to extend from $\mathbb{Z}_p(X)$ to a larger inductive domain V_i by assigning $v_i(\phi_i(X)) = \rho_i$.

• Suppose $\rho = \infty$. In this case, we consider commensurate inductive extensions obtained from V by assigning larger and larger values to ϕ. Then as the values of ϕ approach infinity the valuation domains will converge to an infinite limit domain V_∞. We then extend a single branch from the vertex corresponding to V to a vertex corresponding to V_∞. Limit domains are maximal and cannot be extended. It follows that V_∞ is in T_D and so this corresponds to an endpoint for the branch.

• If the value ρ is finite, by Proposition 2.20, there are two possible cases:

(a) there is just one minimal valuation overring V of D centered on $\mathfrak{m}$ (i.e. $S_{\mathbb{Z}_p(X),\mathfrak{m}}$ consists of one element) and this is exactly V or a Case 3 upside-down domain associated with V (ρ can be rational or irrational);

(b) V is not an overring of D (i.e. $V \notin S_{\mathbb{Z}_p(X),\mathfrak{m}}$). Then, all the elements of T_D centered on $\mathfrak{m}$ are $\succ_{Mac} V$ (ρ is rational).

In both cases we extend $\mathbb{Z}_p(X)$ with V, that is we put V as vertex upon $\mathbb{Z}_p(X)$. Since $V \succeq_{Mac} \mathbb{Z}_p(X)$, this operation does preserve MacLane ordering.

In case (a) we don't extend V further and V becomes the maximal point of this branch of $\mathcal{T}_D$. In fact, $V \in T_D$ and no elements of T_D can be greater than V (because of their minimality as overrings of D). In case (b) we extend V repeating the argument used for extending $\mathbb{Z}_p(X)$. So we will consider the sets S_{V,M_i}, for $i = 1, \cdots, n$, where the M_i's are again the finitely many centers in D_V of the domains of T_D, and they indicate the finitely many directions along which we extend V to the next level. This process can then be extended as often as necessary. Each of these branches will lead to one or more domains of T_D.

If $W_i \in T_D$ is an inductive (commensurable or incommensurable), finite limit or a Case 3 upside-down valuation domain then, looking at the branch defining W_i, at any level we are in the situation described in cases (a) or (b).

Thus we have constructed the desired tree. We observe that the middle vertexes are inductive commensurable domains.

It is easy to see that this same construction still works if we take as a starting point any commensurable domain V such that $W \succeq_{Mac} V$, for each $W \in T_D$, instead of $\mathbb{Z}_p(X)$.

In the above paragraphs we started from a domain D and then indicated a way to represent the domains of T_D as the maximal vertexes of a locally finite tree . Conversely, suppose that we are given a locally finite tree $\mathcal{T}$ whose nonmaximal vertexes are commensurable valuation domains, and the maximal vertexes are inductive (commensurable or incommensurable), limit or Case 3 valuation domains, ordered under MacLane ($\preceq_{Mac}$). The first question we pose is whether the maximal vertexes of $\mathcal{T}$ are minimal (under MacLane ordering) valuation overrings of a certain domain D. We answer this question in the next Theorem 3.2.

Lemma 3.1. *Suppose we are given finitely many inductive (commensurable or incommensurable), limit or Case* 3 *upside-down valuation domains $V_1, \cdots, V_n \in T_p$, and set $D := V_1 \cap \cdots \cap V_n \cap \mathbb{Q}[X]$. If P is a p-unitary prime ideal of D, then P contains exactly one of the centers P_i of the V_i's in D.*

Proof. We observe that the center of V_i in D (namely, P_i) is the contraction in D of the valuation prime of D_{V_i}. Moreover, by Lemma 2.2, the valuation prime of D_{V_i} is the radical of p. Assume that P does not contain any P_i, for $i = 1, \cdots, n$. Thus, setting $S := D \backslash P$, we have that $\mathbb{Q}[X] \subseteq S^{-1} D_{V_i}$, whence $\mathbb{Q}[X] \subseteq D_P$, against the assumption that P is p-unitary. So P contains at least one P_i, for $i = 1, \cdots, n$.

In order to see that P contains exactly one P_i, without loss of generality, we may assume that $n = 2$. So let $D = V_1 \cap V_2 \cap \mathbb{Q}[X]$. We also suppose that $V_1 \not\succeq_{Mac} V_2$ and that $V_2 \not\succeq_{Mac} V_1$, otherwise $D = V_i \cap \mathbb{Q}[X]$, for some $i = 1, 2$, and the thesis would immediately follow (again from the fact that the valuation prime in D_{V_i} is the radical of p). We claim that there exists an inductive commensurable valuation domain V such that $V_i \succeq_{Mac} V$, for $i = 1, 2$, and V_1 and V_2 have different centers in D_V. In fact, consider the centers of V_1 and V_2 in $\mathbb{Z}_p[X]$. If they are distinct, we put $V := \mathbb{Z}_p(X)$. If V_1 and V_2 have the same center M in $\mathbb{Z}_p[X]$, then M is height-two (otherwise $M = p\mathbb{Z}_p[X]$ and $V_1 = V_2 = \mathbb{Z}_p(X)$). We extend $\mathbb{Z}_p(X)$ using a key polynomial ϕ associated with M and distance $\rho := \min\{v_1(\phi), v_2(\phi)\}$. We get a new commensurable valuation domain W_1 such that $V_i \succeq_{Mac} W_1$, $i = 1, 2$. We apply to W_1 the same argument used for $\mathbb{Z}_p(X)$ and find a commensurable valuation domain W_2. After finitely many steps we will find the desired commensurable valuation domain. If not, we would have an infinite ascending sequence of commensurable valuation domains $\{W_k\}_{k\geq 0}$ such that $V_i \succeq_{Mac} W_k$, for each $k \geq 0$ and $i = 1, 2$. Thus $V_1 = V_2 = \lim_{k\to\infty} W_k$.

Now, consider the ring D_V. Then $D_V \subseteq D$. Suppose that $P_1 \subseteq P$ and $P_2 \subseteq P$ and let $\mathfrak{m}_1 := P_1 \cap D_V$ and $\mathfrak{m}_2 := P_2 \cap D_V$. Then, $\mathfrak{m}_1 \subseteq P \cap D_V$ and $\mathfrak{m}_2 \subseteq P \cap D_V$. But, by construction, $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are distinct height-two primes in D_V, so $D_V \subseteq P$ and this is a contradiction. □

A collection of nonzero integral ideals $\mathcal{S}$ in D is a *generalized multiplicative system* if the product of two ideals in $\mathcal{S}$ is in $\mathcal{S}$. The *generalized quotient ring of D with respect to $\mathcal{S}$* is defined as follows ([12, § 4]):

$$D_{\mathcal{S}} := \{x \in K \mid xI \subseteq D, \text{ for some } I \in \mathcal{S}\}.$$

Any quotient ring or intersection of quotient rings of D is a generalized quotient ring of D. In particular, if $\Delta \subseteq \mathrm{Spec}(D)$, then $\bigcap_{P\in\Delta} D_P = D_{\mathcal{S}_\Delta}$, where

$$\mathcal{S}_\Delta = \{\mathcal{I} \mid \mathcal{I} \text{ is an ideal of } \mathcal{D},\ \mathcal{I} \nsubseteq \mathcal{P}, \text{ for each } \mathcal{P} \in \Delta\}.$$

Theorem 3.2. *Assume the notation above and suppose that there is given a collection $\mathcal{V}$ of inductive (commensurable or incommensurable), limit or Case 3 upside-down valuation domains that can be represented as the maximal vertexes of a locally finite tree $\mathcal{T}$. Set $D := \bigcap_{V\in\mathcal{V}} D_V$.*

(a) *If P is a p-unitary prime ideal of D, then $D_P = (D_V)_Q$, for some $V \in \mathcal{V}$ and some p-unitary prime ideal Q of D_V. Also, there exists a p-unitary valuation overring R of D, with maximal ideal M, such that $D_P = (D_R)_{M\cap D_R}$.*

(b) *Let $W \in T_D$ and let P be the center of W in D. Then $D_P = (D_W)_Q$, for some p-unitary prime ideal Q of D_W. Moreover,*

$D_W = D_P \cap \mathbb{Q}[X] = D_{\mathcal{S}_\Delta}$, where Δ is the subset of $\mathrm{Spec}(D)$ *consisting of P and all the upper to zero primes of D.*

(c) *Each $V \in \mathcal{V}$ is a p-unitary valuation overring of D minimal under MacLane ordering (i.e. $V \in T_D$).*

Proof. (a) For each vertex V of $\mathcal{T}$, let $\mathrm{Ht}(V)$ to denote the level of $\mathcal{T}$ at which V appears (that is, the length of the branch defining V) and set $\mathcal{T}(k) := \{V \in \mathcal{V} \mid \mathrm{Ht}(V) < k\}$. By the locally finite character of $\mathcal{T}$, we have that $\mathcal{T}(k)$ is a finite set.

For each $k < \infty$, consider all the valuation domains (vertexes) of $\mathcal{T}$ belonging to level k, say $\{V_{k,1}, \cdots, V_{k,r_k}\}$, and the valuation domains in $\mathcal{T}(k)$. Intersect all them with $\mathbb{Q}[X]$ and set:

$$D_k := V_{k,1} \cap \cdots \cap V_{k,r_k} \cap (\bigcap_{V \in \mathcal{T}(k)} V) \cap \mathbb{Q}[X].$$

By construction, for each domain $V \in \mathcal{V}$ such that $\mathrm{Ht}(V) \geq k$ there exists a valuation domain $V_{k,j}$ (for $j = 1, \cdots, r_k$) such that $V \succeq_{Mac} V_{k,j}$ (hence $D_V \supseteq D_{V_{k,j}}$). Thus:

$$D = \bigcap_{V \in \mathcal{V}} D_V \supseteq \bigcap_{j=1,\cdots r_k} D_{V_{k,j}} \cap \bigcap_{V \in \mathcal{T}(k)} D_V = D_k$$

Let $P_k := P \cap D_k$: this is a p-unitary prime ideal of D_k. Set $S := D \backslash P$ and $S_k := D_k \backslash P_k$ (in particular, $S_k = S \cap D_k$). Then:

$$\begin{aligned} D_P &\supseteq S^{-1} \bigcup_{k \geq 0} D_k = \bigcup_{k \geq 0} S_k^{-1} D_k \\ &= \bigcup_{k \geq 0} (\bigcap_{j=1,\cdots,r_k} S_k^{-1} D_{V_{k,j}} \cap \bigcap_{V \in \mathcal{T}(k)} S_k^{-1} D_V). \end{aligned}$$

By Lemma 3.1, P_k contains exactly one of the centers in D of the $V_{k,j}$'s, for $j = 1, \cdots, r_k$, or of the V's, for $V \in \mathcal{T}(k)$. This means that all but one of the domains $S_k^{-1} D_{V_{k,j}}$, for $j = 1, \cdots, r_k$, $S_k^{-1} D_V$, for $V \in \mathcal{T}(k)$, contain $\mathbb{Q}[X]$, and so $S_k^{-1} D_k$ is equal to exactly one of them.

Suppose that, for some $k > 0$, P_k contains the center of some $V \in \mathcal{V}$. Then $S_k^{-1} D_V \subseteq D_P$ and so:

$$D_P \subseteq S^{-1}(S_k^{-1}) D_V = S^{-1} D_V \subseteq S^{-1} D_P = D_P.$$

Thus, $D_P = S^{-1} D_V$.

Conversely, suppose that, for any $k > 0$, P_k contains the center of some $V_{k,j}$, namely V_{k,j_P}, such that $V_{k,j_P} \notin \mathcal{V}$, for each $k \geq 0$. Thus V_{k,j_P} extends to the upper level $k+1$ of $\mathcal{T}$. We claim that $V_{k+1,j_P} \succeq_{Mac} V_{k,j_P}$. If not, arguing as in the proof of Lemma 3.1, we can construct a commensurable valuation domain V', such that the centers of V_{k,j_P} and V_{k+1,j_P} in $D_{V'}$ are distinct height-two primes. By our hypotheses on P, V_{k,j_P} and V_{k+1,j_P}, it would follow that these centers are both contained in P, which is an evident contradiction since P is a proper ideal of D.

Now, let V be the limit of the sequence $\{V_{k,j_P}\}_{k \geq 0}$ (that is, the maximal vertex of the branch defined by the sequence $\{V_{k,j_P}\}_{k \geq 0}$). Then $V \in \mathcal{V}$ and (see Lemma 1.19):

$$D_P \subseteq S^{-1} D_V = S^{-1} \bigcup_{k \geq 0} D_{V_{k,j_P}} = S^{-1} \bigcup_{k \geq 0} S_k^{-1} D_{V_{k,j_P}} \subseteq D_P.$$

So, in both cases, $D_P = S^{-1} D_V$. Hence there exists a p-unitary prime ideal Q in D_V such that $D_P = (D_V)_Q$.

Now, if V is an inductive incommensurable, limit or Case 3 upside-down valuation domain, the only p-unitary prime ideal of D_V is the valuation prime $M \cap D_V$, where M is the maximal ideal of V (Lemma 1.16, Corollaries 1.21 and 1.25 , Corollary 1.33). Hence $Q = M \cap D_V$.

Assume that V is a commensurable valuation domain. Then, if Q is height-one, Q is the valuation prime in D_V and we have done. If Q is height-two, then (page 10) Q corresponds to an irreducible polynomial $f \in F[Y]$, where $F(Y) \cong V/M$. Consider the Case 1 or 2 upside-down valuation domain R constructed as follows:

$$\begin{array}{ccc} R & \longrightarrow & k \backsimeq F[Y]_{f(Y)} \\ \big\downarrow & & \big\downarrow \\ V & \longrightarrow & L \backsimeq F(Y) \end{array}$$

Then, $D_R = D_V$ (Proposition 1.31) and Q is the center of R in D_R (see page 25). Moreover, by pullback properties,

$$(D_R)_{M_R \cap D_R} = (D_V)_Q = D_P,$$

where M_R is the maximal ideal of R.

(b) Since P is a p-unitary prime ideal of D, by point (a) there exists $V \in \mathcal{V}$ such that $D_P = (D_V)_Q$, where Q is a p-unitary prime ideal of D_V. We have that:

$$D_V \subseteq (D_V)_Q \cap \mathbb{Q}[X] \subseteq W \cap \mathbb{Q}[X] = D_W.$$

Again, since V is not a Case 1 or 2 upside-down valuation domain, by Lemma 2.3 $V \preceq_{Mac} W$. For the minimality of W, we have that $V = W$. Hence, $D_P = (D_W)_Q$.

Then, $D_W \subseteq D_P \cap \mathbb{Q}[X] \subseteq W \cap \mathbb{Q}[X] = D_W$. So $D_W = D_P \cap \mathbb{Q}[X]$. Since $\mathbb{Q}[X] = \bigcap_{\mathfrak{q} \in \mathrm{Spec}(D),\, \mathfrak{q} \cap \mathbb{Z} = (0)} D_{\mathfrak{q}}$, we have that $D_W = D_{\mathcal{S}_\Delta}$, where Δ is defined in the statement.

(c) Suppose that $V \notin T_D$. Then, there exists $W \in T_D$ such that $W \prec_{Mac} V$. Let Q be the center of V in D, whence Q is a p-unitary prime ideal of D. By point (a) $D_Q = (D_{V'})_{Q'}$, for some $V' \in \mathcal{V}$ with a p-unitary prime ideal Q'. Then $D_{V'} \subseteq D_V$, whence $V' \preceq_{Mac} V$, by Lemma 2.3. Since V' and V are not MacLane comparable, it follows that $V = V'$. Let M and N be, respectively, the maximal ideals of V and W. Then $N \cap \mathbb{Q}[X] \subseteq M \cap \mathbb{Q}[X]$ and so $N \cap D \subseteq M \cap D = Q$. Thus:

$$(D_V)_{Q'} = D_Q = D_{M \cap D} \subseteq D_{N \cap D} \subseteq (D_W)_{N \cap D_W}.$$

It follows that $D_V \subseteq D_W$ and, again by Lemma 2.3, we have that $V \preceq_{Mac} W$, against the assumption. $\square$

With the previous hypotheses and notation we can write $D = \bigcap_{W \in T_D} W \cap \mathbb{Q}[X] = \bigcap_{W \in T_D} D_W$. We will show that the localization of D in a prime ideal P corresponds to the localization of some component D_W. We need the following lemma.

Corollary 3.3. *Suppose we are given an integrally closed domain D lying between $\mathbb{Z}_p[X]$ and $\mathbb{Q}[X]$. Then, if P is a prime ideal of D, there exist $W \in T_D$ and a prime ideal Q of D_W, such that $D_P = (D_W)_Q$.*

Proof. It is enough to observe that the collection $\mathcal{T}_D$ satisfies the hypothesis of Theorem 3.2. $\square$

4. Prüfer domains. In this paragraph we investigate when a domain D lying between $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$ is Prüfer.

We recall the following well known fact ([12, Theorem 26.1]): "If D is a Prüfer domain, then any overring of D is a Prüfer domain". So if D is a Prüfer domain, each p-unitary component D_p is Prüfer. On the contrary, suppose that each D_p is Prüfer and take a prime ideal Q of D. If $Q \cap \mathbb{Z} = (0)$, then D_Q is a localization of $\mathbb{Q}[X]$ and so it is a valuation domain. If $Q \cap \mathbb{Z} = (q)$, then $D_Q = (D_q)_{Q^e}$, where Q^e is the extension of Q in D_q ([12, Corollary 5.3]). But $(D_q)_{Q^e}$ is a valuation domain since D_q is Prüfer. Thus D_Q is a valuation domain and D is Prüfer.

It follows that establishing whether D is Prüfer is equivalent to studying this property for the p-unitary components D_p. Hence, we fix a prime integer p and analyze this question for domains D lying between $\mathbb{Z}_p[X]$ and $\mathbb{Q}[X]$.

First, we consider the simplest case of domains of the type $D_V = V \cap \mathbb{Q}[X]$, where $V \in T_p$.

Proposition 4.1 *Suppose we are given a valuation domain $V \in T_p$. Then, D_V is a Prüfer domain if and only if V is a limit valuation domain.*

Proof. A domain in T_p can be of the following types: INDUCTIVE COMMENSURABLE, INDUCTIVE INCOMMENSURABLE, UPSIDE-DOWN OR LIMIT.

If V is commensurable, then D_V is a two-dimensional, Noetherian domain (see page 10). Thus, D_V is not Prüfer, since a Prüfer, Noetherian domain is Dedekind ([12, Theorem 37.1]).

Assume that V is incommensurable and P is the valuation prime in D_V. If D_V is Prüfer, then $(D_V)_P$ is a two-dimensional valuation domain (P has height 2 by Lemma 1.15), whence $(D_V)_P$ is infinite limit or upside-down. Since V is a one-dimensional p-unitary valuation overring of $(D_V)_P$, $(D_V)_P$ is upside-down (see page 20). Then V would be commensurable against the assumption.

If V is upside-down and Q is the height-one prime, V_Q is commensurable. Thus, $D_V \subseteq V_Q \cap \mathbb{Q}[X]$ and D_V is not Prüfer, since its overring $V_Q \cap \mathbb{Q}[X]$ is not Prüfer.

Finally, if V is limit, by Corollaries 1.22 and 1.26, D_V is Prüfer. ◻

Theorem 4.2. *Suppose we are given a domain D lying between $\mathbb{Z}_p[X]$ and $\mathbb{Q}[X]$. Then D is Prüfer if and only if D is integrally closed and all the p-unitary valuation overrings of D are limit.*

Proof. Suppose that D is Prüfer. It is straightforward that D is integrally closed. If W is a p-unitary valuation overring of D, then $D \subseteq D_W$. Hence D_W is Prüfer and, by Proposition 4.1, W is limit.

Conversely, assume that all the p-unitary valuation overrings of D are limit and let P be a prime ideal of D. If $P \cap \mathbb{Z} = (0)$, then D_P is a localization of $\mathbb{Q}[X]$, whence it is a valuation domain. If $p \in P$, by Theorem 3.2, $D_P = (D_V)_{M \cap D_V}$, for some p-unitary valuation overring V (with maximal ideal M) of D. By hypothesis V is limit, hence $(D_V)_{M \cap D_V}$ is a valuation domain (Proposition 4.1). Thus D is Prüfer. ◻

Since the limit valuation domains are maximal elements in T_p with respect to $\preceq_{Mac}$ (Lemma 2.10), we conclude that if T_D contains only limit domains, then all the p-unitary valuation overrings of D are limit. Hence we have the following:

Corollary 4.3. *Suppose we are given a domain D lying between $\mathbb{Z}_p[X]$ and $\mathbb{Q}[X]$. Then D is Prüfer if and only if D is integrally closed and all the valuation domains in T_D are limit.*

Now, suppose that D is a domain lying between $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$. It is immediate to see that the p-unitary valuation overrings of D are exactly the p-unitary valuation overrings of D_p. Moreover we know that D is Prüfer if and only if D_p is Prüfer, for each $p \in \mathbb{Z}$. So we can easily globalize the previous results as follows.

Corollary 4.4. *Suppose we are given a domain D lying between $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$. Then D is Prüfer if and only if D is integrally closed and all the p-unitary valuation overrings of D (equivalently, all the domains in T_{D_p}) are limit, for each prime integer p.*

The problem we are going to consider in the following is related to the construction of "new" Prüfer domains D between $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$. For the previous discussion, without loss of generality we can consider domains D lying between $\mathbb{Z}_p[X]$ and $\mathbb{Q}[X]$.

Theorem 4.5. *Suppose we have a locally finite tree $\mathcal{T}$ of inductive domains of T_p, such that every branch of $\mathcal{T}$ converges to a limit valuation domain. Let $\mathcal{W} := \{W_i\}_{i\in I}$ be the collection of these limit domains. Then, $D := \bigcap_{i\in I} W_i \cap \mathbb{Q}[X]$ is Prüfer.*

Proof. Let P be a prime ideal of D. If $P \cap \mathbb{Z} = (0)$, then D_P is a localization of $\mathbb{Q}[X]$, whence it is a valuation domain.

Assume that $p \in P$. For each $k \geq 0$, let $V_{k,1}, \cdots, V_{k,r_k}$ be the valuation domains lying at the level k of $\mathcal{T}$. Now, for each $i \in I$, $W_i = \lim_{k\to\infty} V_{k,i}$, where $V_{k,i} \in \{V_{k,1}, \cdots, V_{k,r_k}\}$. Whence (using Lemma 1.19) we have that:

$$D = \bigcap_{i\in I} W_i \cap \mathbb{Q}[X] = \bigcap_{i\in I}(\bigcup_{k\geq 0}(V_{k,i} \cap \mathbb{Q}[X])) = \bigcup_{k\geq 0}(\bigcap_{j=1}^{r_k} V_{k,j} \cap \mathbb{Q}[X]).$$

By exactly the same argument used in the proof of Theorem 3.2, we can prove that there exist a domain $W \in \mathcal{W}$ and a p-unitary prime ideal Q of D_W such that $D_P = (D_W)_Q$. Since D_W is Prüfer (Proposition 4.1), D_P is a valuation domain. Thus D is Prüfer. ◻

Remark 4.6. The results in this section can be used as a powerful tool to construct interesting examples of Prüfer domains. Care needs to be exercised in the construction however. In particular, it is not always true that if we are given a family of limit valuation domains $\mathcal{W} = \{W_i\}_{i \in I}$, all of which are correspond to maximal vertices of a locally finite tree then the domain $D := \bigcap_{i \in I} W_i \cap \mathbb{Q}[X]$ is Prüfer. The problem is that all of the maximal vertices of the tree need to be limit valuation domains in order for D to be a Prüfer domain and it may happen that any locally finite tree which includes all of the W_i's as maximal vertices also has maximal vertices which correspond to valuation domains which are not limit.

The following example can make this fact clear.

Suppose $V_0 = \mathbb{Z}_p(X)$ and extend it with two valuation domains $V_{1,1}$ and $V_{1,2}$. Suppose that from $V_{1,1}$ there is only one branch spreading out, constructed by using the same key polynomial at any level with increasing values to infinity. Thus, we have that on the top of this branch there is an infinite limit valuation domain.

Then consider $V_{1,2}$. First suppose that $V_{1,2}$ is an extension of $\mathbb{Z}_p(X)$ obtained with a key polynomial ϕ assuming a rational value μ_1. We extend $V_{1,2}$ with two domains $V_{2,1}$ and $V_{2,2}$. The domain $V_{2,2}$ can be any commensurable extension of $V_{1,2}$. As regards $V_{2,1}$, we construct it using ϕ with a rational value $\mu_2 > \mu_1$.

Then, as done for $V_{1,1}$, we can suppose that from $V_{2,2}$ there is only one branch spreading out, leading to an infinite limit. On the other hand, we extend $V_{2,1}$ with two domains $V_{3,1}$ and $V_{3,2}$. We repeat these arguments at any level and we suppose also that $\lim_{k\to\infty} \mu_k = l \in \mathbb{Q}$. Thus, we have that from each domain $V_{k,2}$ comes out only one infinite branch leading to an infinite limit domain. As concerns the domains $V_{k,1}$, they form a sequence converging to a Case 3 upside-down domain, W.

Thus, we get a tree whose maximal points are all infinite limit valuation domains, namely a set $\mathcal{W} = \{W_i\}_{i \in I}$, except one, the one corresponding to the branch $\{V_{k,1}\}_{k \geq 0}$. Thus if $D := \bigcap_{i \in I} W_i \cap \mathbb{Q}[X]$, then $W \in T_D$ though $W \notin \mathcal{W}$. What happens is that when we construct the tree defining the family of limits $\mathcal{W}$, we do get a locally finite tree, but on the other hand we get an extra branch, that we had not considered before, and which converges to a non-limit domain. This new domain W is an ultrafilter limit of the family $\mathcal{W}$.

Note that the problem here centered on the existence of an infinite number of extensions involving larger and larger values assigned to the same key polynomial but without having those values diverge to infinity. If a tree is built in which we use new key polynomials to extend at each stage or we take care to have the values go to infinity when we do use one key polynomial infinitely many times then the result will be a Prüfer domain.

We now give an example which we consider illuminating of how the tree construction described in Section 3 works out. (In this particular case D is a Prüfer domain.)

Examples 4.7. (a) Consider the integer-valued polynomial ring $\mathrm{Int}(\mathbb{Z})$. We know that it is a Prüfer domain ([5, Theorem VI. 1.7]), hence its p-components $\mathrm{Int}(\mathbb{Z})_p = \mathrm{Int}(\mathbb{Z}_p)$ are Prüfer domains. For what we have seen before, the p-unitary valuation overrings of $\mathrm{Int}(\mathbb{Z}_p)$ are limit of sequences of commensurable domains whose defining tree is locally finite.

If $W \in T_p$ is a valuation overring of $\mathrm{Int}(\mathbb{Z}_p)$, then $W = \mathrm{Int}(\mathbb{Z}_p)_{\mathcal{M}_\alpha}$, where $\alpha \in \hat{\mathbb{Z}_p}$ ($\hat{\mathbb{Z}_p}$ is the p-adic completion of $\mathbb{Z}$), and $\mathcal{M}_\alpha := \{f \in \mathrm{Int}(\mathbb{Z}_p) \mid f(\alpha) \in p\hat{\mathbb{Z}_p}\}$ ([5, Proposition V.2.7]). Set $V_\alpha := \mathrm{Int}(\mathbb{Z}_p)_{\mathcal{M}_\alpha}$. We recall that $V_\alpha = \{\varphi \in \mathbb{Q}(X) \mid \varphi(\alpha) \in \hat{\mathbb{Z}_p}\}$ ([22, Lemma 1.1]).

If $\alpha \in \mathbb{Z}_p$, then V_α is the limit of a sequence of commensurable domains constructed using, at any step $k < \infty$, the (same) key polynomial $\phi(X) := X - \alpha$ and any sequence of (rational) values $\{\mu_k\}_{k\geq 0}$ converging to ∞. We discuss the details.

First, consider $\alpha = 0$. So, for each $k \geq 0$, $\phi_k(X) = X$, $\mu_k = k$, and V_k is the commensurable domain extended from V_{k-1} using X with value k. Let V_∞ be the limit of the sequence $\{V_k\}_{k\geq 0}$. If $f(X) = \sum_{i=1}^{s} a_i X^i \in \mathbb{Q}[X]$, we have that $v_k(f) = \min_{i=1,\cdots,s}\{v_p(a_i) + ki\}$, where v_p is the p-adic valuation on $\mathbb{Q}$. Hence, for $k >> 0$, $v_k(f) = v_p(a_0)$ and $v_\infty(f) = v_p(a_0)$. So, $v_\infty(f) \geq 0$ if and only if $v_p(a_0) \geq 0$, that is $a_0 \in \mathbb{Z}_p$. Since $a_0 = f(0)$, we have that $f \in V_\infty$ if and only if $f \in V_0$. If we take a rational function $\varphi = f/g$, with $f, g \in \mathbb{Q}[X]$, then $v_\infty(\varphi) = v_\infty(f) - v_\infty(g) = v_p(a_0) - v_p(b_0) = v_p(a_0/b_0) = v_p(\varphi(0))$. So we have that $\varphi \in V_\infty$ if and only if $\varphi(0) \in \mathbb{Z}_p$ if and only if $f \in V_0$ (observe that if $\alpha \in \mathbb{Z}_p$ and $\varphi \in \mathbb{Q}(X)$, then $\varphi(\alpha) \in \mathbb{Z}_p$ if and only if $\varphi(\alpha) \in \hat{\mathbb{Z}_p}$). Hence $V_\infty = V_0$.

For the case $0 \neq \alpha \in \mathbb{Z}_p$, it is enough to observe that there is an isomorphism $\gamma : V_0 \to V_\alpha, \quad X \mapsto X - \alpha$. So it is enough to choose $\phi_k(X) = X - \alpha$ and proceeding exactly as for $\alpha = 0$.

More generally, suppose that $\alpha \in \hat{\mathbb{Z}_p} \backslash \mathbb{Z}_p$. Then $\alpha = \sum_{i \geq 0} a_i p^i$, where $a_i \in \mathbb{Z}$, $0 \leq a_i \leq p-1$. We construct V_k extending V_{k-1} by using $\phi_k := X - (a_0 + a_1 p + \cdots + a_k p^k)$ with value $\mu_k := k$. By the same arguments used above, it is easy to check that V_α is the limit of the sequence $\{V_k\}_{k \geq 0}$.

For simplicity of notation, we write that W is extended from V using (ϕ, μ) if W is a first stage extension of V accomplished using the key polynomial ϕ with value μ. We construct the tree as follows.

Level 0. The basis is $\mathbb{Z}_p(X)$.

Level 1. Extend $\mathbb{Z}_p(X)$ to $V_{1,1}$ using $(X, 1)$, to $V_{1,2}$ using $(X-1, 1)$, to $V_{1,3}$ using $(X-2, 1)$, $\cdots$, to $V_{1,p}$ using $(X-(p-1), 1)$.

Level 2. Extend $V_{1,1}$ only once using $(X, 2)$. Extend $V_{1,2}, \cdots, V_{1,p}$ p times each, in the following way: if $V_{1,j}$ $(2 \leq j \leq p)$ is extended from $V_{1,1}$ using $(X - (j-1), 1)$, then the extensions of $V_{1,j}$ to level 2 are made using $(X - ((j-1) + tp), 2)$, with $t = 1, \cdots, p$.

Level k+1. Take V at level k which is extended from V' (at level $k-1$). If V is constructed using the same key polynomial ϕ used to construct V', then we extend V only once using $(\phi, k+1)$. If V is constructed using a key polynomial $\overline{\phi} = X - (a_0 + a_1 p + a_2 p^2 + \cdots + a_{k-1} p^{k-1})$ different from ϕ, we extend V p times using:

$$(\overline{\phi}_t = X - (a_0 + a_1 p + a_2 p^2 + \cdots + a_{k-1} p^{k-1} + t p^k), k+1),$$

$t = 1, \cdots, p$. In this way we obtain a sequence $\{V_n\}_{n \geq 0}$, where V_n is extended from V_{n-1} using the key polynomial $X - (a_0 + a_1 p + \cdots + a_{n-1} p^{n-1})$ with value n. The limit of this sequence is V_α, with $\alpha = \sum_{i \geq 0} a_i p^i$. Thus, we get all and only the p-unitary valuation overrings of $\mathrm{Int}(\mathbb{Z}_p)$.

In particular, take $p = 2$. Using the above construction, the tree $\mathcal{T}_{\mathrm{Int}(\mathbb{Z}_2)}$ is the following:

(b) Consider the previous example of $\mathrm{Int}(\mathbb{Z}_p)$. The key polynomials used in the construction of the tree $\mathcal{T}_{\mathrm{Int}(\mathbb{Z}_p)}$ all have degree 1 and

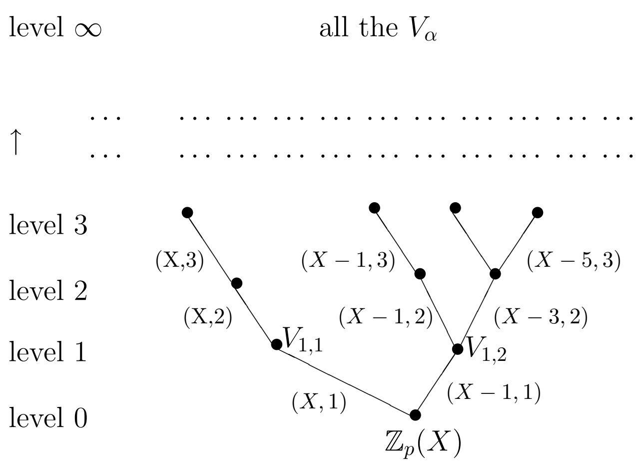

Figure 1

the values assigned to these key polynomials are all integers. The effects of these two types of restrictive choices can be seen in the structure of $\mathrm{Int}(\mathbb{Z}_p)$ in that given any p-unitary maximal ideal, the residue field has order p and the maximal ideal is locally generated by p. MacLane's scheme for building valuation domains allows us to choose key polynomials of larger and larger degrees as we progress up the tree and to choose rational values for these polynomials with larger and larger denominators. Choosing key polynomials of larger degrees leads to larger residue fields and making the value group larger puts the prime p in higher powers of maximal ideals. By not putting a bound on these parameters we can, if we wish, construct

• a limit valuation domain with an infinite (even algebraically closed) residue field.

• a limit valuation domain with an infinitely generated value group (even the entire additive group of rational numbers) - in particular, the maximal ideal will not be finitely generated.

The observations above address the motivation for this paper. The rings $\mathrm{Int}(\mathbb{Z}_p)$ can seem like very esoteric examples because their behavior is so unlike that of $\mathbb{Z}_p[X]$ and $\mathbb{Q}[X]$. We can use the results of this paper however, to produce a wide variety of Prüfer domains with uncountably many p-unitary maximal ideals and, as noted above, with more freedom as regards residue fields and value groups.

5. Integrally closed domains. In this section we classify some relevant classes of integrally closed domains D lying between $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$ through their representation as an intersection of given valuation domains.

As we pointed out in the introduction, every overring of $\mathbb{Z}[X]$ is known when its p-components D_p are known, for each prime integer p. Moreover, D is integrally closed if and only if D_p is integrally closed, for all $p \in \mathbb{Z}$. Hence, we focus our attention on the integrally closed domains lying between $\mathbb{Z}_p[X]$ and $\mathbb{Q}[X]$ and then try to generalize the obtained results to rings lying between $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$. We soon point out that things do not always go as well as in the Prüfer case, that is many properties (for instance, the Noetherian property) may be satisfied by all the p-components of a domain D, but not by D itself. Thus, in many cases, we will have a quite precise characterization of the domains between $\mathbb{Z}_p[X]$ and $\mathbb{Q}[X]$, which cannot be extended to domains between $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$.

We follow Notation 2.9 and use T_D to denote the set of the valuation domains in T_p which are minimal with respect to being overrings of D under $\preceq_{Mac}$.

Theorem 5.1. *Suppose we are given an integrally closed domain D lying between $\mathbb{Z}_p[X]$ and $\mathbb{Q}[X]$. Then D is Krull if and only if $D = V_1 \cap V_2 \cap \cdots \cap V_n \cap \mathbb{Q}[X]$, where $V_i \in T_p$ is inductive commensurable or finite limit. In particular D is Noetherian and it is the intersection of a semi-local Principal Ideal Domain with $\mathbb{Q}[X]$.*

Proof. Suppose that D is Krull. Then D is a locally finite intersection of DVR's: say $D = \bigcap_{i\in I} V_i$. Because of finite character of the intersection, there are only finitely many domains V_i in which p is noninvertible. Hence $D = V_1 \cap V_2 \cap \cdots \cap V_n \cap \mathbb{Q}[X]$, where $V_1, \cdots, V_n \in T_p$. A p-unitary valuation domain V_i is a DVR if and only if it is commensurable or finite limit. By Theorem 0.1 D is Noetherian. Moreover, $V_1 \cap V_2 \cap \cdots \cap V_n$ is a semi-local Principal Ideal Domain ([12, § 37, Ex. 4]).

Conversely, if $D = V_1 \cap V_2 \cap \cdots \cap V_n \cap \mathbb{Q}[X]$, where each V_i is commensurable or finite limit, then D is Krull because it is a finite intersection of Krull domains ([12, Corollary 44.10]). $\square$

We recall that for a nonzero (fractional) ideal I of a domain D the *divisorial closure of* I is $I_v := (D\colon (D\colon I))$. The ideal I is 1*divisorial* when $I = I_v$.

An ideal J of D is a *Glaz-Vasconcelos ideal* (in short, a GV-ideal) if J is finitely generated and $(D\colon J) = D$ (i.e. $J_v = D$). The set of Glaz-Vasconcelos ideals of D is denoted by $\mathrm{GV}(D)$. Then for a nonzero (fractional) ideal I of D the *w-closure* of I is defined as follows:

$$I_w := \{x \in K \mid xJ \subseteq I \text{ for some } J \in \mathrm{GV}(D)\}.$$

The ideal I is called a *w-ideal* if $I = I_w$. A *Strong-Mori domain* is a domain that verifies the ascending chain condition on integral w-ideals (see, for instance, [7] and [8]). Strong-Mori domains are a very good generalization of Noetherian domains since they satisfy, in a weaker form, many important properties of Noetherian domains (for instance, if D is Strong-Mori then $D[X]$ is also Strong-Mori and in a Strong-Mori domain the primary decomposition property for w-ideals holds).

When $\dim(D) = 1$, Strong-Mori is always equivalent to Noetherian ([8, Corollary 1.10]). In our context, we can generalize Theorem 5.1 as follows:

Theorem 5.2. *Suppose we are given an integrally closed domain D lying between $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$. The following conditions are equivalent:*

(i) *D is Noetherian;*

(ii) *D is Strong-Mori;*

(iii) *D is Krull.*

Proof. By [8, Theorem 2.8], we have that (ii) $\Leftrightarrow$ (iii).

It is well known that (i) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i) follows from the fact that a Krull overring of a two-dimensional Noetherian domain is Noetherian ([15]). $\square$

Example 5.3. For each fixed prime number $p \in \mathbb{Z}$, consider a fixed limit valuation domain V_p for which the maximal ideal M_p satisfies $M_p \cap \mathbb{Z}[X] = (p, X)$ (to get such a domain it is sufficient to consider an infinite sequence of inductive commensurable domains $\{V_k\}_{k \geq 0}$, converging to a finite limit and such that V_1 is constructed from $\mathbb{Z}_p(X)$ by using a key-polynomial corresponding to the maximal ideal (p, X). Then $V_p \cap \mathbb{Q}[X]$ is a Dedekind domain (Theorem 0.1). Take $D := \bigcap_{p \in \mathbb{Z}} V_p \cap \mathbb{Q}[X]$. Now, $D_p = V_p \cap \mathbb{Q}[X]$, so it is Noetherian, but D is not Noetherian because X is contained in infinitely many height-one prime ideals of D (all the ideals $M_p \cap D$).

We recall that a *Mori domain* is a domain that satisfies the ascending chain condition on integral divisorial ideals (for a survey about Mori domains we suggest, for instance, [3]). Since, for each nonzero fractional ideal I of D, we have $I \subseteq I_w \subseteq I_v$, then a divisorial ideal is always a w-ideal. Thus, Mori domains contain the class of Strong Mori domains.

The following example shows that, in our context, Mori domains represent a larger class of domains than the "Krull - Strong Mori - Noetherian" ones.

Example 5.4. Suppose V is a Case 3 upside-down valuation domain with associated pseudo-valuation domain A (Theorem 1.34). Since A shares its prime spectrum with a DVR, then A is Mori from [3, Theorem 2.2]. Hence, $A \cap \mathbb{Q}[X]$ is Mori because it is the intersection of two Mori domains ([3, Theorem 2.4]). But $A \cap \mathbb{Q}[X]$ is not Noetherian by Corollary 1.33.

We have the following result.

Proposition 5.5. *Suppose we are given a valuation domain $V \in T_p$. Then D_V is Mori if and only if V is inductive commensurable, upside-down or finite limit.*

Proof. If V is commensurable or finite limit, then V is a DVR, hence it is Mori. Thus, D_V is Mori from [3, Theorem 2.4].

Suppose V is upside-down. If V is Case 1 or 2, then $D_V = D_W$, for some commensurable domain W, and it is Mori (since it is Noetherian). If V is Case 3, then $D_V = A \cap \mathbb{Q}[X]$ where A is a pseudo-valuation domain. So D_V is Mori by the using the argument of Example 5.4.

Suppose that V is incommensurable. Then D_V is completely integrally closed ([12, § 13, Ex. 11]). If D_V is Mori, then it is Krull ([3, Theorem 2.3]). But D_V is not Krull since V is not a DVR. So D_V is not Mori.

Finally, if V is infinite limit, then D_V is Prüfer by Proposition 4.1. Suppose that D_V is also Mori. Let D_m be the set of the divisorial maximal ideals of D_V. Then $D_V = \bigcap_{Q \in D_m} (D_V)_Q$ ([3, Theorem 3.3]). But D_V contains only one p-unitary prime ideal, which is the valuation prime P, and it is maximal. Thus $P \in D_m$. Since $V = (D_V)_P$ is not a DVR, by [3, Theorem 3.4] P is strong, that is $PP^{-1} = P$. But P is a maximal, divisorial ideal of a Prüfer domain, hence it is invertible by [10, Corollary 3.1.3]. Then P cannot be strong seeing as $PP^{-1} = D_V$. So D_V is not Mori. □

Theorem 5.6. *Suppose we are given an integrally closed domain D lying between $\mathbb{Z}_p[X]$ and $\mathbb{Q}[X]$. Then D is Mori if and only if $D = V_1 \cap \cdots \cap V_n \cap \mathbb{Q}[X]$, where $V_i \in T_p$ is inductive commensurable, upside-down or finite limit.*

Proof. Suppose that $D = V_1 \cap \cdots \cap V_n \cap \mathbb{Q}[X]$, where the V_i's are as in the statement. By Proposition 5.5, D_{V_i} is Mori, hence $D = \bigcap_{i=1,\cdots,n} D_{V_i}$ is Mori by [3, Theorem 2.4].

Conversely, suppose that D is Mori. We have that $D = \bigcap_{V \in T_D} D_V$. If $V \in T_D$, then D_V is a generalized quotient ring of D (Theorem 3.2). From [3, Theorem 2.5], if D is Mori then D_V is Mori. Thus, by Proposition 5.5, all the elements in T_D (and, consequently, the p-unitary valuation overrings of D) are commensurable, upside-down or finite limit. Let D_m be the set of the divisorial maximal ideals of D. Then $D = \bigcap_{P \in D_m} D_P$ and this intersection has finite character, which means that each nonzero element $x \in D$ is noninvertible in finitely many domains D_P, for $P \in D_m$ ([3, Theorem 3.3]). It follows that there exist only finitely many ideals in D_m containing p, say $P_1, \cdots, P_n$. Thus, $D = D_{P_1} \cap \cdots \cap D_{P_n} \cap \mathbb{Q}[X]$. By Theorem 3.2, $D_{P_i} = (D_{V_i})_{M_i \cap \mathbb{Q}[X]}$, for some p-unitary valuation overring V_i (with maximal ideal M_i) of D. So, on one hand we have that $D \subseteq V_1 \cap \cdots \cap V_n \cap \mathbb{Q}[X]$, since the V_i are overrings of D. On the other hand, we have that:

$$\begin{aligned} V_1 \cap \cdots \cap V_n \cap \mathbb{Q}[X] &\subseteq D_{V_1} \cap \cdots \cap D_{V_n} \cap \mathbb{Q}[X] \\ &\subseteq (D_{V_1})_{M_1 \cap \mathbb{Q}[X]} \cap \cdots \cap (D_{V_n})_{M_n \cap \mathbb{Q}[X]} \\ &\subseteq D. \end{aligned}$$

Thus $D = V_1 \cap \cdots \cap V_n \cap \mathbb{Q}[X]$, where each V_i is inductive commensurable, upside-down or finite limit. □

Corollary 5.7. *Suppose we are given an integrally closed domain D lying between $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$. If D is Mori, then each p-component D_p has the structure given by Theorem* 5.6.

Proof If D is Mori, then D_p is a generalized quotient ring of D and it is Mori from [3, Theorem 2.5] and the thesis follows. □

The converse of Corollary 5.7 is not true, since Example 5.7 shows that D may have each p-component which is Mori (actually, D_p is Dedekind), but D itself is not Mori. In fact, since D is completely integrally closed ([12, § 13, Ex. 11]), if D is Mori, then it is Krull ([3, Theorem 2.3]) and, by Theorem 5.2, D is Noetherian. But Example 5.3 says that D is not Noetherian.

We recall that the *t-closure* of a nonzero fractional ideal I in a domain D is:

$$I_t := \bigcup \{J_v \mid J \subseteq I,\ J \text{ finitely generated}\},$$

and I is a t-ideal if $I = I_t$. Moreover, I is t-prime if I is a t-ideal and it is also prime. A *Prüfer v-multiplication domains* (PvMD) is a domain such that its localizations at each t-prime ideal is a valuation domain. Thus, PvMD's naturally generalize Prüfer domains (for which this last property holds for all prime ideals). For a more detailed account about PvMD's we refer the reader to [14, 17, 18, 20]. We just mention that PvMD's are tightly linked to GCD domains (i.e. domains for which every pair of both nonzero elements has a greatest common divisor). In fact a domain D is a GCD-domain if and only if it is a PvMD with zero t-class group ([4]). A PvMD is an integrally closed domain since it is an intersection of valuation domains.

Theorem 5.8. *Suppose we are given an integrally closed domain D lying between $\mathbb{Z}_p[X]$ and $\mathbb{Q}[X]$. Then, if D is a PvMD each $V \in T_D$ is inductive commensurable or limit. The converse holds if the intersection $D = \bigcap_{V \in T_D} D_V$ has finite character (i.e. each nonzero element $x \in D$ is noninvertible in just finitely many domains D_V, and this is equivalent to ask that x is contained in finitely many centers of the V's in D).*

Proof. Suppose that D is a PvMD. If $V \in T_D$ is incommensurable or a Case 3 upside-down and M is the radical of p, then $M \cap D_V$ is the only p-unitary prime ideal of D_V (Lemma 1.3, Lemma 1.16 and Lemma 1.32). So $M \cap D_V$ is minimal over p and the same is true for $(M \cap D_V)(D_V)_{M \cap D_V}$, which is then a t-ideal from [Corollaire 3, p.31]. Let $P := M \cap D$. By Theorem 3.2, since $M \cap D_V$ is the only p-unitary prime ideal of D_V, we have that $D_P = (D_V)_{M \cap D_V}$. So $PD_P = (M \cap D_V)(D_V)_{M \cap D_V}$ and P is a t-ideal in D by [17, Lemma 3.17]. Then D_P is a valuation domain. But $D_P = (D_V)_{M \cap D_V}$ is not a valuation domain. In fact, if V is incommensurable, by Corollary 1.18 $(D_V)_{M \cap D_V} \neq V$ and, by the minimality of V, it cannot be another valuation domain. If V is Case 3 upside-down, then $D_V = A \cap \mathbb{Q}[X]$, for some pseudo-valuation domain $A \subset V$ (Theorem 1.33). Then $D_P \subseteq A \neq V$ and, again by the minimality of V, D_P cannot be a valuation domain. Thus, each $V \in T_D$ is commensurable or limit.

Conversely, suppose that each $V \in T_D$ satisfies the hypotheses of the statement. Let P be a t-prime ideal of D. If $P \cap \mathbb{Z} = (0)$, then D_P is a localization of $\mathbb{Q}[X]$, whence it is a valuation domain. If $p \in P$, then $D_P = (D_V)_Q$, for some $V \in T_D$ and p-unitary prime ideal Q of D_V (Theorem 3.2).

If V is limit, then the only p-unitary prime ideal of D_V is $M \cap D_V$, where M is the maximal ideal of V. In this case $D_P = (D_V)_{M \cap D_V} = V$ (Lemma 1.3 and Proposition 4.1), so it is a valuation domain. If V is commensurable then $D_P = (D_V)_Q$, where Q can be either the center of V, and in this case $(D_V)_Q = V$ is a valuation domain; or Q is a height-two p-unitary prime, and then $D_P = (D_V)_Q$ is not a valuation domain (since it is a two-dimensional Noetherian domain). We want to show that, in this last case, P is not a t-prime ideal of D. The domain D_P is two-dimensional, Noetherian in which P_V is height-one and P is height-two. Put $PD_P := (a_1, \cdots, a_n)\frac{1}{s}$, where $a_1, \cdots, a_n \in D$

and $s \in D\backslash P$, and $J := (p, a_1, \cdots, a_n)$. Then $P_V \subsetneq J \subseteq P$. From the assumed finite character on the V's in T_D, modulo adding finitely many elements to J, we can also say that $J \nsubseteq P_W$, for each $W \in T_D$, and $J \nsubseteq \mathfrak{q}$, where $\mathfrak{q}$ runs among the upper to zero primes of D (in this last case, it is sufficient to add the prime number p

It is well-known that $J^{-1} \subseteq \bigcap_{\mathfrak{p} \in \mathrm{Spec}(D),\, J \nsubseteq \mathfrak{p}} D_{\mathfrak{p}}$ (the second term of the equation is called the *Kaplansky Transform* of J). Thus we have that $J^{-1} \subseteq \bigcap_{W \in T_D} (D_{P_W}) \cap \mathbb{Q}[X] = \bigcap_{W \in T_D} D_W = D$. Thus $J_v = D$ hence $P_t = D$ and P is not t-prime.

It follows that D is a PvMD. $\square$

We recall that a *Krull-type domain* is a domain D which can be realized as an intersection of the following type

$$D = \bigcap_{P \in \mathcal{P}} D_P,$$

where $\mathcal{P} \subseteq \mathrm{Spec}(D)$, D_P is a valuation domain and the intersection is locally finite. For example, *Krull domains* or *generalized Krull domains* ([12, § 43]) are Krull-type domains. In [14] the author shows that Krull-type domains are exactly the PvMD's with t-finite character (i.e. each nonzero element $x \in D$ belongs to finitely many t-prime ideals).

Corollary 5.9. *Suppose we are given an integrally closed domain D lying between $\mathbb{Z}_p[X]$ and $\mathbb{Q}[X]$. Then D is Krull-type if and only if $D = V_1 \cap \cdots \cap V_n \cap \mathbb{Q}[X]$, where $V_i \in T_p$ is inductive commensurable or limit.*

Proof. If D is Krull-type, then it is a PvMD and, by Theorem 5.8, $V \in T_D$ is inductive commensurable or limit. Take $V \in T_D$ with maximal ideal M. Let $\mathfrak{q}_V := M \cap D$ and $Q := M \cap D_V$ be the centers of V in D and in D_V. Thus, $D_V = D_{\mathfrak{q}_V} \cap \mathbb{Q}[X]$ and $\mathfrak{q}_V D_{\mathfrak{q}_V} = Q(D_V)_Q$. Now, Q is minimal over p, for any $V \in T_D$ (Q is the radical of p). So $\mathfrak{q}_V D_{\mathfrak{q}_V}$ is minimal over p, and it is a t-ideal in $D_{\mathfrak{q}_V}$ ([16, Corollaire 3, p.31]). By [17, lemma 3.17], $\mathfrak{q}_V$ is a t-ideal in D. Hence the centers of the $V \in T_D$ are t-ideals of D. We now show that they are distinct. By Theorem 3.2, we have that $D_{\mathfrak{q}_V} = (D_V)_Q$. If V is commensurable or limit, then $(D_V)_Q = V$ (Lemma 1.3 and Proposition 4.1). So, if

$V \in T_D$, the localization of D at the center of V is V itself. Our claim follows. Since p is contained in all these centers, from the t-finite character of D it follows T_D is finite and so D satisfies the the statement as being $D = \bigcap_{V \in T_D} D_V = \bigcap_{V \in T_D} V \cap \mathbb{Q}[X]$.

Conversely, suppose that $D = V_1 \cap \cdots \cap V_n \cap \mathbb{Q}[X]$, where $V_i \in T_p$ is commensurable or limit and, without loss of generality, the V_i's are not MacLane comparable among themselves. We claim that $T_D = \{V_1, \cdots, V_n\}$. If W is a p-unitary valuation overring of D with maximal ideal N, then $N \cap D$ is a p-unitary prime ideal of D. By Lemma 3.1, P contains the center P_i of exactly one V_i. Set $S := D \backslash P$. Then, $W = S^{-1}W \supseteq S^{-1}D = S^{-1}D_{V_i}$. Thus, $W \succeq_{Mac} V_i$ and the claim follows. Since V_i is commensurable or limit, following the proof of Theorem 3.2, we have that V_i is a localization of D. Thus D is Krull-type. $\square$

In general, if $\mathbb{Z}[X] \subseteq D \subseteq \mathbb{Q}[X]$, we have that if D is Krull-type then D_p is Krull-type, for each $p \in \mathbb{Z}$, and so it has the structure given in this last Corollary. But, conversely, if D_p is Krull-type then D is not necessarily Krull-type. Again Example 5.3 shows that such a situation may occur. In fact, D_p is Dedekind (so Krull-type), but D is not Krull-type since the element X is contained in infinitely many t-maximal ideals (by construction, for each $p \in \mathbb{Z}$, there exists a p-unitary t-maximal ideal containing X).

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