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Mi Hee Park · Francesca Tartarone

Polynomial closure in essential domains

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Abstract. Let D be a domain, E a subset of its quotient field K, and $\mathrm{Int}(E, D) = \{f \in K[X] \mid f(E) \subseteq D\}$. The *polynomial closure* of E is the set $\mathrm{cl}_D(E) = \{x \in K \mid f(x) \in D, \forall f \in \mathrm{Int}(E, D)\}$. We compare the polynomial closure with the divisorial closure in a general setting and then in an essential domain. Especially, we show that these two closures of ideals are the same if D is a Krull-type domain.

Introduction

Let D be an integral domain with quotient field K. For each subset $E \subseteq K$, $\mathrm{Int}(E, D) := \{f \in K[X] \mid f(E) \subseteq D\}$ is called the *ring of D-integer-valued polynomials over E*. As usual, when $E = D$, we set $\mathrm{Int}(D) := \mathrm{Int}(D, D)$.

The *polynomial closure* (in D) of E is defined as the set

$$\mathrm{cl}_D(E) := \{x \in K \mid f(x) \in D, \forall f \in \mathrm{Int}(E, D)\},$$

that is, $\mathrm{cl}_D(E)$ is the largest subset $F \subseteq K$ such that $\mathrm{Int}(E, D) = \mathrm{Int}(F, D)$.

The first papers about polynomial closure (for instance, [3], [8], [10] and [12]) set it in a topological context. Among many other results, it was proven that in a Dedekind domain with finite residue fields, the polynomial closure of a subset E is the same as the intersection of its topological closures in every maximal ideal-adic topology.

There is another way to look at the polynomial closure. That is to study it as a star-operation [7, Lemma 1.2]. Recall that a fractional ideal I of D is a D-module such that $dI \subseteq D$ for some nonzero element $d \in D$, and denote by $\mathfrak{F}(D)$ the set of nonzero fractional ideals of D. A star-operation is a mapping $* : \mathfrak{F}(D) \longrightarrow \mathfrak{F}(D)$, $I \mapsto I^*$, satisfying the following properties for each $a \in K\backslash\{0\}$ and $I, J \in \mathfrak{F}(D)$:

$(*1)$ $(aD)^* = aD$; $(aI)^* = aI^*$;
$(*2)$ $I \subseteq I^*$;

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M.H. Park: Department of Mathematics Chung-Ang University Seoul 156-756, Korea. e-mail: mhpark@cau.ac.kr

F. Tartarone: Dipartimento di Matematica Università degli studi Roma Tre Largo San Leonardo Murialdo 1, 00146 Roma, Italy. e-mail: tfrance@mat.uniroma3.it

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($*3$) $I \subseteq J \Rightarrow I^* \subseteq J^*$;
($*4$) $(I^*)^* = I^*$.

The best known star-operation is the v-closure (or, the divisorial closure):

$$I \mapsto I_v := (I^{-1})^{-1},$$

where $I^{-1} := \{x \in K \mid xI \subseteq D\}$. A nonzero fractional ideal I is called *divisorial* if $I = I_v$. The divisorial closure has a prominent position in this context; it is the maximal star-operation in the sense that $I^* \subseteq I_v$ for each $I \in \mathfrak{F}(D)$ and each star-operation $*$.

In Section 1, we point out some interesting analogies between the polynomial closure and the divisorial closure. We observe that the divisorial closure of a fractional ideal I can be defined in the same way as the polynomial closure is defined just replacing the set of polynomials $\mathrm{Int}(I, D)$ by $I^{-1}X$.

Motivated by this observation, we introduce a new star-operation, $\mathrm{Cl}_D(-)$, which is constructed like the polynomial closure but using the set of polynomials $D[X/I] := \bigcap_{a \in I\setminus\{0\}} D[\frac{X}{a}]$ instead of $\mathrm{Int}(I, D)$. When $D[X/I] = \mathrm{Int}(I, D)$ (which happens, for example, whenever $\mathrm{Int}(D) = D[X]$), obviously $\mathrm{Cl}_D(I) = \mathrm{cl}_D(I)$. In general, $I^{-1}X \subseteq D[X/I] \subseteq \mathrm{Int}(I, D)$, whence $\mathrm{cl}_D(I) \subseteq \mathrm{Cl}_D(I) \subseteq I_v$.

In Section 2, we prove that $\mathrm{Cl}_D(-)$ is equal to the divisorial closure in any essential domains (Proposition 2.4). Moreover, $\mathrm{Cl}_D(-)$ has turned out to be an useful tool to study the equivalence between the polynomial closure and the divisorial closure in this class of domains. The study about the polynomial closure in this direction was developed by M. Fontana *et al.* in [7]. They proved, for instance, that in a valuation domain the polynomial closure and the divisorial closure of a fractional ideal are the same. We prove, more generally, that these two closures coincide in a large class of essential domains, the Krull-type domains, which include Krull domains and semi-quasi-local Prüfer domains (Theorem 2.8).

1. A new star-operation $\mathrm{Cl}_D(-)$

Let I be a nonzero fractional ideal of D and define $I^{-1} = \{x \in K \mid xI \subseteq D\}$. Every D-homomorphism from I to D can be uniquely extended to a D-homomorphism from K to K. Henceforth, we identify $\mathrm{Hom}_D(I, D)$ with the subset of $\mathrm{Hom}_D(K, K)$ mapping I into D. For each $f \in \mathrm{Hom}_D(K, K)$, if we let $a = f(1)$, then f is the multiplication on K by a. Thus we have a canonical isomorphism of D-modules

$$\varphi : I^{-1} \longrightarrow \mathrm{Hom}_D(I, D)$$

defined by $\varphi(a)(x) = ax$ for all $a \in I^{-1}$ and $x \in I$. Note that each function $\varphi(a)$ is a polynomial function induced by the polynomial aX. By identifying I^{-1} with $\mathrm{Hom}_D(I, D)$, we also have a canonical isomorphism

$$\lambda : I_v \longrightarrow \mathrm{Hom}_D(\mathrm{Hom}_D(I, D), D)$$

defined by $\lambda(x)(f) = f(x)$ for all $x \in I_v$ and $f \in \mathrm{Hom}_D(I, D)$ [11, page 37].

Thus we have $I_v = \{x \in K \mid f(x) \in D, \forall f \in \mathrm{Hom}_D(I, D)\} = \{x \in K \mid f(x) \in D, \forall f \in I^{-1}X\}$. Recalling that $\mathrm{cl}_D(I) = \{x \in K \mid f(x) \in D, \forall f \in \mathrm{Int}(I, D)\}$, we can see that the divisorial closure I_v is a sort of polynomial closure obtained by substituting $I^{-1}X$ for $\mathrm{Int}(I, D)$. From the inclusion $I^{-1}X \subseteq \mathrm{Int}(I, D)$, it directly follows the fact that $I_v \supseteq \mathrm{cl}_D(I)$ [7, Corollary 1.3 (4)].

Another interesting analogy between the two types of closure is that as $\mathrm{cl}_D(I)$ is the largest fractional ideal of D such that $\mathrm{Int}(I, D) = \mathrm{Int}(\mathrm{cl}_D(I), D)$, so I_v is the largest fractional ideal of D such that $\mathrm{Hom}_D(I, D) = \mathrm{Hom}_D(I_v, D)$.

Borrowing the above idea, that is, using a suitable subset of polynomials, we will construct a new star operation later in this section. To begin with, we recall the following result:

Lemma 1.1. [6, Lemma 4.5] *Let D be a domain such that* $\mathrm{Int}(D) = D[X]$. *Then, for each nonzero D-submodule E of K, we have*

$$\mathrm{Int}(E, D) = D[X/E] := \bigcap_{a \in E\setminus\{0\}} D[\frac{X}{a}].$$

Moreover, $D[X/E]$ is a graded ring of the following type

$$D[X/E] = D \oplus (\bigcap_{u \in E\setminus\{0\}} \frac{1}{u}D)X \oplus \cdots \oplus (\bigcap_{u \in E\setminus\{0\}} \frac{1}{u^n}D)X^n \oplus \cdots .$$

Let I be a nonzero fractional ideal of a domain D. We let $I(n)$ denote the D-module generated by the set $\{u^n \mid u \in I\}$.

Proposition 1.2. *Let D be a domain such that* $\mathrm{Int}(D) = D[X]$. *For $I, J \in \mathfrak{F}(D)$, $\mathrm{cl}_D(I) = \mathrm{cl}_D(J)$ if and only if $I(n)_v = J(n)_v$ for each $n \geq 0$.*

Proof. From Lemma 1.1, $\mathrm{Int}(I, D)$ is a graded ring of the form

$$\bigoplus_{n \geq 0} (\bigcap_{u \in I\setminus\{0\}} \frac{1}{u^n}D)X^n.$$

Note that

$$\bigcap_{u \in I\setminus\{0\}} \frac{1}{u^n}D = I(n)^{-1}.$$

The same holds for $\mathrm{Int}(J, D)$. Thus, $\mathrm{Int}(I, D) = \mathrm{Int}(J, D)$ if and only if $I(n)^{-1} = J(n)^{-1}$, that is, $I(n)_v = J(n)_v$ for each $n \geq 0$. □

Remark 1.3. Let D be a domain such that $\mathrm{Int}(D) = D[X]$. If I is a nonzero fractional ideal of D, then by Lemma 1.1 and [1, Corollary 1, II.1.6], we have the canonical isomorphisms

$$\begin{aligned}\operatorname{Hom}_D(\operatorname{Int}(I,D),D) &= \operatorname{Hom}_D(D[X/I],D)\\ &= \operatorname{Hom}_D(D\oplus I^{-1}X\oplus\cdots\oplus I(n)^{-1}X^n\oplus\cdots,D)\\ &\cong \prod_{n\geq 0}\operatorname{Hom}_D(I(n)^{-1}X^n,D)\\ &\cong \prod_{n\geq 0}\operatorname{Hom}_D(I(n)^{-1},D)\\ &\cong \prod_{n\geq 0} I(n)_v.\end{aligned}$$

Let $\theta : \operatorname{Hom}_D(\operatorname{Int}(I,D),D)\longrightarrow \prod_{n\geq 0} I(n)_v$ be the isomorphism defined by the above series of natural isomorphisms and let $\psi : \mathrm{cl}_D(I)\longrightarrow \operatorname{Hom}_D(\operatorname{Int}(I,D),D)$ be the injective map defined by $\psi(x)(f) = f(x)$ for all $f\in\operatorname{Int}(I,D)$. Then the composite map $\theta\psi : \mathrm{cl}_D(I)\longrightarrow\prod_{n\geq 0} I(n)_v$ is injective and it is given by $x\mapsto (1,x,x^2,x^3,\ldots)$. (For the better understanding, we describe the isomorphism θ^{-1} specifically: Let $\overline{x} = (x_0,x_1,x_2,\ldots)\in\prod_{n\geq 0} I(n)_v$. Then $\theta^{-1}(\overline{x})$ is given by $f=\sum_{i=0}^n a_iX^i\mapsto\sum_{i=0}^n a_ix_i$.)

Proposition 1.4. *Let D be a domain such that* $\operatorname{Int}(D) = D[X]$. *If I is a nonzero fractional ideal of D, then*

$$\mathrm{cl}_D(I) = \{x\in K \mid x^n\in I(n)_v,\ \forall n\geq 0\}.$$

Proof. It follows from the map $\theta\psi$ mentioned above. Here is a more elementary proof.

Let $x\in\mathrm{cl}_D(I)$. Since $I(n)^{-1}X^n\in\operatorname{Int}(I,D) = D[X/I]$, by the definition of polynomial closure, $I(n)^{-1}x^n\in D$. Thus $x^n\in I(n)_v$ for each $n\geq 0$.

Conversely, let $x\in K$ such that $x^n\in I(n)_v$ for each $n\geq 0$. Then $I(n)^{-1}x^n\subseteq D$. So, for an arbitrary polynomial $f(X) = a_0+a_1X+\cdots+a_nX^n\in\operatorname{Int}(I,D) = D[X/I] = \bigoplus_{n\geq 0} I(n)^{-1}X^n$, we have $f(x) = a_0+a_1x+\cdots+a_nx^n\in D$. Thus $x\in\mathrm{cl}_D(I)$. □

Motivated by Proposition 1.4, we define a new star-operation without the assumption that $\operatorname{Int}(D) = D[X]$.

Definition 1.5. *For a nonzero fractional ideal I of D, we set*

$$\begin{aligned}\mathrm{Cl}_D(I) &:= \{x\in K\mid x^n\in I(n)_v,\ \forall n\geq 0\}\\ &= \{x\in K\mid f(x)\in D,\ \forall f\in D[X/I]\}.\end{aligned}$$

Lemma 1.6. *With the above notation, we have*

(1) *the map* $\mathrm{Cl}_D(-) : \mathfrak{F}(D)\to\mathfrak{F}(D)$, $I\mapsto\mathrm{Cl}_D(I)$, *is a star-operation;*
(2) *for each $I\in\mathfrak{F}(D)$,* $\mathrm{cl}_D(I)\subseteq\mathrm{Cl}_D(I)$ *(the equality holds when* $\operatorname{Int}(D) = D[X]$*).*

Proof. (1) It is an easy exercise to check that all the properties required for a star-operation are satisfied.

(2) Since $D[X/I]\subseteq\operatorname{Int}(I,D)$, it is obvious that $\mathrm{cl}_D(I)\subseteq\mathrm{Cl}_D(I)$. In particular, if $\operatorname{Int}(D) = D[X]$, then $\operatorname{Int}(I,D) = D[X/I]$ by Lemma 1.1 and hence $\mathrm{cl}_D(I) = \mathrm{Cl}_D(I)$. □

2. Polynomial closure and divisorial closure in essential domains

Let D be an integral domain with $D = \bigcap_{P \in \mathcal{P}} D_P$ for some subset $\mathcal{P} \subseteq \mathrm{Spec}(D)$. The *$\mathcal{P}$-polynomial closure of $I \in \mathfrak{F}(D)$* is

$$\mathcal{P}\text{-}\mathrm{cl}_D(I) := \bigcap_{P \in \mathcal{P}} \mathrm{cl}_{D_P}(ID_P).$$

The mapping $I \mapsto \mathcal{P}\text{-}\mathrm{cl}_D(I)$ defines a star-operation on D with $\mathcal{P}\text{-}\mathrm{cl}_D(I) \subseteq \mathrm{cl}_D(I)$ [7, Lemma 1.4].

In [7] M. Fontana *et al.* study the polynomial closure in essential domains and compare it with other star-operations including the one just mentioned. In this section, we carry on this studying.

We recall some relevant definitions. An *essential domain* is a domain D such that

$$D = \bigcap_{P \in \mathcal{P}} D_P, \tag{2.1}$$

where $\mathcal{P} \subseteq \mathrm{Spec}(D)$ and each D_P is a valuation domain. A domain D is *Krull-type* if it is essential and the intersection (2.1) is locally finite (that is, each nonzero element $x \in K$ is a unit in D_P for all but finitely many $P \in \mathcal{P}$), and a Krull-type domain is *strong Krull-type* if the valuation domains D_P are pairwise independent (that is, each pair of valuation domains D_P doesn't have common overrings except K) [9, §43].

By [7, Theorem 1.9], if D is an essential domain with the representation (2.1), then we have

$$I_t \subseteq \mathcal{P}\text{-}\mathrm{cl}_D(I) \subseteq \mathrm{cl}_D(I) \subseteq I_v$$

for each nonzero fractional ideal I. If D is strong Krull-type, then $\mathcal{P}\text{-}\mathrm{cl}_D(I) = \mathrm{cl}_D(I)$ for each $I \in \mathfrak{F}(D)$ and under some conditions on the set of t-maximal ideals of D, $\mathrm{cl}_D(I) = I_v$ [7, Corollary 1.12]. We strengthen these results in the following.

Remark 2.1. We will freely use the following facts:

(a) Let D be a domain, I an ideal of D and A an overring of D. Then, $(A\colon I) = (A\colon IA)$.

(b) If $D = \bigcap_{\lambda \in \Lambda} D_\lambda$, where D_λ are overrings of D, and I is a nonzero fractional ideal of D, then

$$(D\colon I) = \bigcap_{\lambda \in \Lambda} (D_\lambda\colon I) = \bigcap_{\lambda \in \Lambda} (D_\lambda\colon ID_\lambda).$$

(c) Let V be a valuation domain. Then
- each prime ideal $P \subseteq V$ is *divided*, that is, $P = PV_Q$ for each prime ideal Q of V such that $P \subseteq Q$ [9, Theorem 17.6 (b)];
- a nonzero ideal I of V is invertible if and only if it is principal [9, Theorem 17.1 (1)]; moreover, if I is a prime ideal, then it is maximal [9, Theorem 17.3 (a)].

Theorem 2.2. *Let $D = \bigcap_{P \in \mathcal{P}} D_P$, $\mathcal{P} \subseteq \mathrm{Spec}(D)$, be a Krull-type domain. Then, $\mathcal{P}\text{-cl}_D(I) = I_v$ for all $I \in \mathfrak{F}(D)$ if and only if D is a strong Krull-type domain.*

Proof. Let I be a nonzero integral ideal of the Krull-type domain D. We claim first that

$$(D\colon I)D_Q = \bigcap_{P \in \mathcal{P}} \big((D_P\colon I)D_Q\big) \tag{2.2}$$

for all $Q \in \mathcal{P}$. The containment $(D : I)D_Q \subseteq \bigcap_{P \in \mathcal{P}} \big((D_P : I)D_Q\big)$ is clear. For the reverse inclusion, take $\xi \in \bigcap_{P \in \mathcal{P}} \big((D_P : I)D_Q\big) \setminus \{0\}$. Then ξ is a unit in D_P for all but finitely many $P \in \mathcal{P}$, say $P_1, \dots, P_r$. Since $\xi \in (D_{P_i} : I)D_Q$, there exists $s_i \in D \setminus Q$ such that $s_i\xi \in (D_{P_i} : I)$. Let $s = s_1 \cdots s_r$. Then $s \in D \setminus Q$ and $s\xi I \subseteq D_{P_i}$ for all $i = 1, \dots, r$. It follows that $s\xi I \subseteq \bigcap_{P \in \mathcal{P}} D_P = D$, whence $s\xi \in (D : I)$ and $\xi \in (D : I)D_Q$.

Now assume that D is a strong Krull-type domain. Since the D_P's are pairwise independent valuation domains, $D_P D_Q = K$ for any two distinct prime ideals P and Q in $\mathcal{P}$. Thus we have

$$\begin{aligned}(D : I)D_Q &= \bigcap_{P \in \mathcal{P}} \big((D_P : I)D_Q\big) = \bigcap_{P \in \mathcal{P}} \big((D_P : I)D_P D_Q\big)\\ &= (D_Q : I) = (D_Q : ID_Q).\end{aligned}$$

Therefore,

$$\begin{aligned}I_v &= (D : (D : I)) = \bigcap_{P \in \mathcal{P}} (D_P : (D : I))\\ &= \bigcap_{P \in \mathcal{P}} (D_P : (D : I)D_P) = \bigcap_{P \in \mathcal{P}} (D_P : (D_P : ID_P))\\ &= \bigcap_{P \in \mathcal{P}} (ID_P)_v = \bigcap_{P \in \mathcal{P}} \mathrm{cl}_{D_P}(ID_P)\\ &= \mathcal{P}\text{-cl}_D(I),\end{aligned}$$

where the last second equality follows from [7, Proposition 1.8].

For the converse, we assume that $\mathcal{P}\text{-cl}_D(I) = I_v$ for each $I \in \mathfrak{F}(D)$ but D is not a strong Krull-type domain. For each distinct prime ideals P, Q in $\mathcal{P}$, let $q(P, Q)$ be the prime ideal of D such that $D_P D_Q = D_{q(P,Q)}$. In other words, $q(P, Q)$ is the prime ideal maximal among all prime ideals contained in $P \cap Q$. Note that $q(P, Q)$ is (0) if and only if D_P and D_Q are independent. Since D is not a strong Krull-type domain, there exists a prime ideal $P_0 \in \mathcal{P}$ such that D_{P_0} and D_Q are dependent for some $Q(\neq P_0) \in \mathcal{P}$. Put $q := \bigcup_{Q \in \mathcal{P}\setminus\{P_0\}} q(P_0, Q)$. Since D_{P_0} is a valuation domain, the set of prime ideals $\{q(P_0, Q) \mid Q \in \mathcal{P} \setminus \{P_0\}\}$ is linearly ordered by inclusion and hence q is a nonzero prime ideal of D contained in P_0. Moreover, since the intersection $D = \bigcap_{P \in \mathcal{P}} D_P$ is locally finite, there exist finitely many prime ideals $P_1, \dots, P_n \in \mathcal{P}$ $(n \geq 1)$ such that $q = q(P_0, P_1) = \cdots = q(P_0, P_n)$ and $q \supsetneq q(P_0, Q)$ for all $Q \in \mathcal{P} \setminus \{P_0, \dots, P_n\}$.

Let a be a nonzero element of q. Then a is contained in only finitely many prime ideals of $\mathcal{P}$, say $P_0, \dots, P_n, P_{n+1}, \dots, P_{n+m}$. Note that $q \not\subseteq P_{n+i}$ for all $i = 1, \dots, m$. Choose $b \in q \setminus \bigcup_{j=1}^m P_{n+j}$ and put $I := (a, b)D_{P_0} \cap D$. Then $ID_{P_0} = (a, b)D_{P_0}$. For $i = 1, \dots, n$, we have

$$ID_{P_i} = (a, b)D_{P_0}D_{P_i} \cap D_{P_i} = (a, b)D_q \cap D_{P_i} = (a, b)D_q,$$

because $(a, b)D_q \subseteq qD_q = qD_{P_i} \subseteq D_{P_i}$. For $i = n+1, \dots, n+m$, $ID_{P_i} = D_{P_i}$, because $b \in I \setminus P_i$. For $P \in \mathcal{P} \setminus \{P_0, \dots, P_{n+m}\}$, $ID_P = D_P$, because $a \in I \setminus P$. Note that since D_{P_0} is a valuation domain, $ID_{P_0} = (a, b)D_{P_0}$ is principal, say cD_{P_0} (where $c = a$ or b) and so $(a, b)D_q = (a, b)D_{P_0}D_q = cD_q$. Then we have

$$\begin{aligned}
I^{-1} &= (D_{P_0} : ID_{P_0}) \cap \left(\bigcap_{i=1}^n (D_{P_i} : ID_{P_i})\right) \cap \left(\bigcap_{P \in \mathcal{P}\setminus\{P_0,\dots,P_n\}} (D_P : ID_P)\right) \\
&= (D_{P_0} : cD_{P_0}) \cap \left(\bigcap_{i=1}^n (D_{P_i} : cD_q)\right) \cap \left(\bigcap_{P \in \mathcal{P}\setminus\{P_0,\dots,P_n\}} (D_P : ID_P)\right) \\
&= \frac{1}{c}D_{P_0} \cap \left(\bigcap_{i=1}^n \frac{1}{c}qD_{P_i}\right) \cap \left(\bigcap_{P \in \mathcal{P}\setminus\{P_0,\dots,P_n\}} D_P\right) \\
&= \frac{1}{c}\left(D_{P_0} \cap \left(\bigcap_{i=1}^n qD_{P_i}\right) \cap \left(\bigcap_{P \in \mathcal{P}\setminus\{P_0,\dots,P_n\}} cD_P\right)\right) \\
&\subseteq \frac{1}{c}\left(qD_q \cap \bigcap_{P \in \mathcal{P}} D_P\right) = \frac{1}{c}q,
\end{aligned}$$

where the last second inclusion follows from the fact $qD_{P_i} = qD_q \subseteq D_{P_i}$, $i = 1, \dots, n$. Hence,

$$cq^{-1} \subseteq I_v = \mathcal{P}\text{-cl}_D(I) \subseteq \mathrm{cl}_{D_{P_0}}(ID_{P_0}) = (ID_{P_0})_v = cD_{P_0}.$$

Thus we have $q^{-1} \subseteq D_{P_0}$, that is, $(D : q)D_{P_0} = D_{P_0}$. Meanwhile, by (2.2),

$$\begin{aligned}
(D : q)D_{P_0} &= \bigcap_{P \in \mathcal{P}} \big((D_P : qD_P)D_{P_0}\big) \\
&= \left(\bigcap_{i=0}^n \big((D_{P_i} : qD_{P_i})D_{P_0}\big)\right) \cap \left(\bigcap_{P \in \mathcal{P}\setminus\{P_0,\dots,P_n\}} \big((D_P : qD_P)D_{P_0}\big)\right) \\
&= \left(\bigcap_{i=0}^n D_qD_{P_0}\right) \cap \left(\bigcap_{P \in \mathcal{P}\setminus\{P_0,\dots,P_n\}} D_PD_{P_0}\right) \\
&= D_q \cap \left(\bigcap_{P \in \mathcal{P}\setminus\{P_0,\dots,P_n\}} D_{q(P,\,P_0)}\right) \\
&= D_q,
\end{aligned}$$

where the third equality follows from [5, Corollary 3.1.5] and [9, Theorem 7.6], and the last equality follows from the fact that $q(P, P_0) \subseteq q$ for all $P \in \mathcal{P} \setminus \{P_0\}$. Thus we have $D_{P_0} = D_q$, which contradicts the fact that $q \subsetneq P_0$. □

Corollary 2.3. *If D is a strong Krull-type domain, then* $\mathrm{cl}_D(I) = I_v$ *for all* $I \in \mathfrak{F}(D)$.

Proposition 2.4. *Let* $D = \bigcap_{P \in \mathcal{P}} D_P$ *be an essential domain. Then* $\mathrm{Cl}_D(I) = I_v$ *for all* $I \in \mathfrak{F}(D)$.

Proof. Let I be a nonzero integral ideal of D. By Proposition 1.2, $\mathrm{Cl}_D(I) = I_v$ if and only if $(I(n))_v = (I_v(n))_v$ for all $n \geq 1$. Put $I^* := \bigcap_{P \in \mathcal{P}} ID_P$. Then the mapping $I \mapsto I^*$ defines a star-operation on D, whence $I \subseteq I^* \subseteq I_v$ and $(I^*)_v = I_v$ [9, Theorem 32.5 and Theorem 34.1].

We claim that $I(n)D_P = I^n D_P$ for all $n \geq 1$ and all $P \in \mathcal{P}$. Let $a_1, \ldots, a_n \in I$. Since D_P is a valuation domain, we may assume that $a_1 D_P \supseteq a_2 D_P \supseteq \cdots \supseteq a_n D_P$. Then $a_1 \cdots a_n \in {a_1}^n D_P \subseteq I(n)D_P$. Therefore $I^n D_P \subseteq I(n)D_P$. The reverse inclusion $I(n)D_P \subseteq I^n D_P$ is obvious.

Therefore,

$$I(n)^* = \bigcap_{P \in \mathcal{P}} I(n)D_P = \bigcap_{P \in \mathcal{P}} I^n D_P = (I^n)^*,$$

and so we have

$$\begin{aligned}(I_v(n))_v \supseteq (I(n))_v &= \big(I(n)^*\big)_v = \big((I^n)^*\big)_v \\ &= (I^n)_v = \big((I_v)^n\big)_v \supseteq (I_v(n))_v\,.\end{aligned}$$

Thus $I(n)_v = (I_v(n))_v$ for all $n \geq 1$ and $\mathrm{Cl}_D(I) = I_v$. □

Corollary 2.5. *Let* $D = \bigcap_{P \in \mathcal{P}} D_P$ *be an essential domain such that* $\mathrm{Int}(D) = D[X]$. *Then* $\mathrm{cl}_D(I) = I_v$ *for all* $I \in \mathfrak{F}(D)$.

Proof. This directly follows from Lemma 1.6 and Proposition 2.4. □

Lemma 2.6. [4, Proposition 1.8 (b) and Corollary 1.9] *Let D be a domain with a nonzero divided prime ideal q. Then the map* $I \mapsto \frac{I}{q}$ *establishes a one-to-one correspondence between the set of all the fractional ideals of D such that* $q \subsetneq I \subseteq I_v \subsetneq D_q$ *and the set of all the nonzero fractional ideals of* $\frac{D}{q}$. *Furthermore, if* $I \in \mathfrak{F}(D)$ *such that* $q \subsetneq I \subseteq I_v \subsetneq D_q$, *then*

$$\left(\frac{I}{q}\right)^{-1} = \frac{I^{-1}}{q} \quad \text{and} \quad \left(\frac{I}{q}\right)_v = \frac{I_v}{q}.$$

Proposition 2.7. *Let D be a semi-quasi-local Prüfer domain. Then* $\mathrm{cl}_D(I) = I_v$ *for all* $I \in \mathfrak{F}(D)$.

Proof. Let $P_1, \dots, P_n$ be the maximal ideals of D. We use the induction on n. If $n = 1$, then D is a valuation domain, so the conclusion follows from [7, Proposition 1.8] (or Corollary 2.3).

Assume that $n > 1$ and that the theorem holds for Prüfer domains with at most $(n-1)$ maximal ideals. By rearrangements, if necessary, we may assume that there exists a positive integer $r (\le n)$ such that D_{P_i} and D_{P_1} are dependent for $i \le r$, and D_{P_i} and D_{P_1} are independent for $i > r$. Put $S := \bigcap_{i=1}^r D_{P_i}$ and $T := \bigcap_{i=r+1}^n D_{P_i}$ (where $T := K$, if $r = n$). So $D = S \cap T$. Note that they are Prüfer domains with r and $n - r$ maximal ideals, respectively [9, Theorem 22.8].

Case 1. Assume that $r < n$. By induction hypothesis, the theorem holds for S and T. So, in particular, for each nonzero integral ideal I of D, $\mathrm{cl}_S(IS) = (IS)_v$ and $\mathrm{cl}_T(IT) = (IT)_v$. Since $S = D_{S_0}$ and $T = D_{T_0}$ for the multiplicative subsets $S_0 = D \setminus \bigcup_{i=1}^r P_i$ and $T_0 = D \setminus \bigcup_{i=r+1}^n P_i$ [9, Lemma 5.4], we have that $\mathrm{cl}_S(IS) = \mathrm{cl}_S(I)$ and $\mathrm{cl}_T(IT) = \mathrm{cl}_T(I)$ [3, Lemma 3.4]. So $(IS)_v = \mathrm{cl}_S(I)$ and $(IT)_v = \mathrm{cl}_T(I)$.

We observe that $ST = K$. Actually, since $q(P_1, P_i) \ne (0)$ for all $i \le r$ and $q(P_1, P_j) = (0)$ for all $j \ge r+1$, we have $q(P_i, P_j) = (0)$ for $i \le r$ and $j \ge r+1$. (Here, we use the same notation as in the proof of Theorem 2.2.) This implies that there does not exist nonzero prime ideals q of D such that $ST \subseteq D_q$, and hence by [9, Theorem 26.1], $ST = K$. Therefore,

$$\begin{aligned}(D\colon I)S &= (D\colon I)D_{S_0} = ((S\colon I) \cap (T\colon I))D_{S_0} \\ &= (S\colon I)D_{S_0} \cap (T\colon I)D_{S_0} = (S\colon I) \cap (T\colon I)TS \\ &= (S\colon I) = (S\colon IS),\end{aligned}$$

and similarly,

$$(D\colon I)T = (T\colon I) = (T\colon IT).$$

Thus we have

$$\begin{aligned}I_v &= (D\colon (D\colon I)) = (S\colon (D\colon I)) \cap (T\colon (D\colon I)) \\ &= (S\colon (D\colon I)S) \cap (T\colon (D\colon I)T) = (S\colon (S\colon IS)) \cap (T\colon (T\colon IT)) \\ &= (IS)_v \cap (IT)_v = \mathrm{cl}_S(I) \cap \mathrm{cl}_T(I) \\ &= \mathrm{cl}_{D_{S_0}}(I) \cap \mathrm{cl}_{D_{T_0}}(I) \subseteq \mathrm{cl}_D(I),\end{aligned}$$

where the last inclusion follows from [2, Lemma IV.2.1]. Hence $I_v = \mathrm{cl}_D(I)$.

Case 2. Assume that $r = n$, so that $q(P_1, P_i) \ne (0)$ for each $i = 2, \dots, n$. Let $q := \bigcap_{i=2}^n q(P_1, P_i)$. Then q is a nonzero prime ideal of D and $q = q(P_1, P_j)$ for some $j \ge 2$. Without loss of generality, we may assume that $q = q(P_1, P_n)$. Since q is contained in all the maximal ideals of D, q is a divided ideal, that is, $qD_q = q$ (in fact, $qD_q = qD_{P_i}$ for all $i = 1, \dots, n$, whence $qD_q = \bigcap_{i=1}^n qD_{P_i} = q$). So, for each nonzero integral ideal I of D, I and q are comparable. In fact, if $I \not\subseteq q$, then $ID_{P_i} \not\subseteq qD_{P_i}$, for all $i = 1, \dots, n$. As D_{P_i} being a valuation domain, $q = qD_{P_i} \subseteq ID_{P_i}$ for all $i = 1, \dots, n$, whence $q \subseteq \bigcap_{i=1}^n ID_{P_i} = I$.

Case 2.1. Assume that $q \subsetneq I$. Then by Lemma 2.6, $(\frac{I}{q})_v = \frac{I_v}{q}$. Note that $\frac{D}{q}$ is a semi-quasi-local Prüfer domain with n-maximal ideals $\frac{P_1}{q}, \cdots, \frac{P_n}{q}$. Moreover,

$(\frac{D}{q})_{\frac{P_1}{q}} \cong \frac{D_{P_1}}{q}$ and $(\frac{D}{q})_{\frac{P_n}{q}} \cong \frac{D_{P_n}}{q}$ are independent. So, by Case 1, $(\frac{I}{q})_v = \mathrm{cl}_{\frac{D}{q}}(\frac{I}{q})$. Since q is nonmaximal, q has infinite residue field, whence $\mathrm{Int}(D) \subseteq D_q[X]$ [2, Proposition I.3.4]. Therefore, by [7, Proposition 3.2], $\mathrm{cl}_{\frac{D}{q}}(\frac{I}{q}) = \frac{\mathrm{cl}_D(I)}{q}$. Thus we have

$$\frac{I_v}{q} = (\frac{I}{q})_v = \mathrm{cl}_{\frac{D}{q}}(\frac{I}{q}) = \frac{\mathrm{cl}_D(I)}{q},$$

and hence $I_v = \mathrm{cl}_D(I)$.

Case 2.2. Assume that $I \subseteq q$ and ID_q is not a principal ideal of D_q, hence it is not invertible. Then $II^{-1} \subseteq ID_q(ID_q)^{-1} \subseteq qD_q = q$. So,

$$(ID_q)^{-1} \subseteq (q\colon ID_q) \subseteq (q\colon I) \subseteq (D : I) = I^{-1} \subseteq (ID_q)^{-1}.$$

Thus we have $(ID_q)^{-1} = (q\colon I) = I^{-1}$. Since $I^n D_q$ is not invertible for all $n \geq 1$, similarly we have

$$(I^n D_q)^{-1} = (D_q\colon I^n D_q) = (D\colon I^n) = (I^n)^{-1}.$$

Then, by the same argument used in the proof of Proposition 2.4, $I(n)D_q = I^n D_q$ and so

$$\begin{aligned} I(n)^{-1} &= (D\colon I(n)) \subseteq (D_q\colon I(n)D_q) = (D_q\colon ID_q(n)) \\ &= (D_q\colon I^n D_q) = (D\colon I^n) = (I^n)^{-1} \subseteq I(n)^{-1}. \end{aligned}$$

Now we claim that $\mathrm{Int}(I, D) = D[X/I]$. By [6, Lemma 4.5] and [3, Lemma 3.4],

$$D[X/I] \subseteq \mathrm{Int}(I, D) \subseteq \mathrm{Int}(I, D_q) = \mathrm{Int}(ID_q, D_q).$$

Since $\mathrm{Int}(D_q) = D_q[X]$, again by [6, Lemma 4.5], $\mathrm{Int}(ID_q, D_q) = D_q[X/ID_q]$. From the above equations, it follows that

$$\begin{aligned} D_q[X/ID_q] &= \bigoplus_{n\geq 0}(ID_q(n))^{-1}X^n \\ &= D_q \oplus \bigoplus_{n\geq 1} I(n)^{-1}X^n \\ &= D_q + D[X/I]. \end{aligned}$$

Thus we have

$$D[X/I] \subseteq \mathrm{Int}(I, D) \subseteq D_q + D[X/I].$$

But, $\mathrm{Int}(I, D) \cap K = D$, so that $\mathrm{Int}(I, D) = D[X/I]$.

Therefore, $\mathrm{cl}_D(I) = \mathrm{Cl}_D(I)$, and hence $\mathrm{cl}_D(I) = I_v$ by Proposition 2.4.

Case 2.3. Assume that $I \subseteq q$, $ID_q = aD_q$ for some $a \in I$, but $I_v \neq aD_q$. Put $J := a^{-1}I$. Then since $D \subseteq J \subseteq J_v \subsetneq D_q$, $\frac{J}{q}$ is a nonzero fractional ideal of $\frac{D}{q}$. By the same argument as in Case 2.1, we have $J_v = \mathrm{cl}_D(J)$, from which it follows that $I_v = \mathrm{cl}_D(I)$.

Case 2.4. Assume that $I \subseteq q$, $ID_q = aD_q$ for some $a \in I$, and $I_v = aD_q$. Put $J := a^{-1}I$. Then $J_v = D_q$, and hence by Lemma 2.6, $\frac{J}{q}$ and $\frac{J_v}{q}$ are not fractional ideals of $\frac{D}{q}$. Since $\frac{D}{q}$ is a Prüfer domain, it is integrally closed and so $\mathrm{Int}(\frac{J}{q}, \frac{D}{q}) = \frac{D}{q} = \mathrm{Int}(\frac{J_v}{q}, \frac{D}{q})$ by [2, Proposition I.1.9]. Moreover,

$$\begin{aligned} q[X] &\subseteq D[X/D_q] \subseteq \mathrm{Int}(D_q, D) = \mathrm{Int}(J_v, D) \subseteq \mathrm{Int}(J, D) \\ &\subseteq \mathrm{Int}(J, D_q) = \mathrm{Int}(JD_q, D_q) = \mathrm{Int}(D_q) = D_q[X]. \end{aligned}$$

Considering the canonical map

$$\Phi : D_q[X] \to (\frac{D_q}{qD_q})[X] = (\frac{D_q}{q})[X],$$

we have

$$\frac{\mathrm{Int}(J_v, D)}{q[X]} = \Phi(\mathrm{Int}(J_v, D)) \cong \mathrm{Int}(\frac{J_v}{q}, \frac{D}{q}) = \frac{D}{q}$$

and

$$\frac{\mathrm{Int}(J, D)}{q[X]} = \Phi(\mathrm{Int}(J, D)) \cong \mathrm{Int}(\frac{J}{q}, \frac{D}{q}) = \frac{D}{q}$$

by [2, Proposition I.3.8 and Remark I.3.9] and [7, Lemma 3.1]. Therefore, $\mathrm{Int}(J_v, D) = D + q[X] = \mathrm{Int}(J, D)$, which implies that $J_v = \mathrm{cl}_D(J)$. Therefore, $I_v = \mathrm{cl}_D(I)$. □

Theorem 2.8. *Let $D = \bigcap_{P \in \mathcal{P}} D_P$ be a Krull-type domain. Then $\mathrm{cl}_D(I) = I_v$ for all $I \in \mathfrak{F}(D)$.*

Proof. Let I be a nonzero proper integral ideal of D. Then I is contained in only finitely many prime ideals $P_1, \dots, P_n$ in $\mathcal{P}$. Put $S := \bigcap_{i=1}^n D_{P_i}$ and $T := \bigcap_{P \in \mathcal{P} \setminus \{P_1, \dots, P_n\}} D_P$. (We may assume that $\mathcal{P} \setminus \{P_1, \dots, P_n\} \neq \emptyset$, otherwise $D = S$ is a semi-quasi-local Prüfer domain, so the conclusion follows from Proposition 2.7.) So $D = S \cap T$.

Notice that $(T : I) = T$. In fact,

$$\begin{aligned} (T : I) &= \bigcap_{P \in \mathcal{P} \setminus \{P_1, \dots, P_n\}} (D_P : I) = \bigcap_{P \in \mathcal{P} \setminus \{P_1, \dots, P_n\}} (D_P : ID_P) \\ &= \bigcap_{P \in \mathcal{P} \setminus \{P_1, \dots, P_n\}} D_P = T. \end{aligned}$$

We claim that $(D : I)S = (S : I)$. Put $S_0 := D \setminus \bigcup_{i=1}^n P_i$. Then $S = D_{S_0}$, and

$$\begin{aligned} (D : I)S &= ((S : I) \cap (T : I))\, S = ((S : I) \cap T)\, D_{S_0} \\ &= (S : I) D_{S_0} \cap T_{S_0} = (S : I) \cap T_{S_0}. \end{aligned}$$

Since $(S : I) \subseteq (ST : I) = (T_{S_0} : I)$, to get the equality $(D : I)S = (S : I)$ it suffices to show that $(T_{S_0} : I) = T_{S_0}$. By the local finiteness of the intersection $T = \bigcap_{P \in \mathcal{P} \setminus \{P_1, \dots, P_n\}} D_P$, it follows that $T_{S_0} = \bigcap_{P \in \mathcal{P} \setminus \{P_1, \dots, P_n\}} (D_P)_{S_0}$ [9, Proposition

43.5]. Moreover, each D_P is a valuation domain, so $(D_P)_{S_0} = D_{q(P)}$ for some prime ideal $q(P)$ contained in P. Note that $I \not\subseteq q(P)$ for all $P \in \mathcal{P} \setminus \{P_1, \dots, P_n\}$. Therefore,

$$(T_{S_0} \colon I) = \bigcap_{P \in \mathcal{P} \setminus \{P_1, \dots, P_n\}} (D_{q(P)} \colon I) = \bigcap_{P \in \mathcal{P} \setminus \{P_1, \dots, P_n\}} D_{q(P)} = T_{S_0}.$$

Thus we have

$$\begin{aligned} I_v &\subseteq I_v S = (D \colon (D \colon I))S \subseteq (S \colon (D \colon I)S) \\ &= (S \colon (S \colon I)) = (IS)_v = \mathrm{cl}_S(IS) = \mathrm{cl}_S(I), \end{aligned}$$

where the last second equality follows from Proposition 2.7. Also, observe that

$$\begin{aligned} \mathrm{Int}(I, T) &= \bigcap_{P \in \mathcal{P} \setminus \{P_1, \dots, P_n\}} \mathrm{Int}(I, D_P) = \bigcap_{P \in \mathcal{P} \setminus \{P_1, \dots, P_n\}} \mathrm{Int}(ID_P, D_P) \\ &= \bigcap_{P \in \mathcal{P} \setminus \{P_1, \dots, P_n\}} \mathrm{Int}(D_P) = \mathrm{Int}(T), \end{aligned}$$

where the last equality follows from [2, Proposition I.2.5]. In particular, this observation implies that $\mathrm{Int}(I, T) = \mathrm{Int}(I_v, T)$, because $\mathrm{Int}(T) \subseteq \mathrm{Int}(I_v, T) \subseteq \mathrm{Int}(I, T)$.

Finally, we get

$$\begin{aligned} \mathrm{Int}(I, D) &= \mathrm{Int}(I, S) \cap \mathrm{Int}(I, T) = \mathrm{Int}(\mathrm{cl}_S(I), S) \cap \mathrm{Int}(I_v, T) \\ &\subseteq \mathrm{Int}(I_v, S) \cap \mathrm{Int}(I_v, T) = \mathrm{Int}(I_v, D) \subseteq \mathrm{Int}(I, D). \end{aligned}$$

Thus $\mathrm{Int}(I, D) = \mathrm{Int}(I_v, D)$, and hence $\mathrm{cl}_D(I) = I_v$. □

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