Geometry of genus 8 Nikulin surfaces and rationality of moduli

In honour of Alberto Collino

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1. Nikulin surfaces in low genus

- Nikulin surface of genus g: a complex K3 surface S endowed with
  - a[1] polarization $\mathcal{C}$ of genus g,
  - a line bundle $\mathcal{M} := \mathcal{O}_S(M)$ so that $2M \sim N$
  - and N is the disjoint union of 8 copies of $\mathbf{P}^1$.
- The irreducible components of the moduli space have dimension 11 and are essentially characterized by $< \mathcal{C}, \mathcal{M} >$. [2]
- Further assumption in this talk:

$$< \mathcal{C}, \mathcal{M} >= 0.$$

This defines an integral component of the moduli, unique for

$$g \equiv 0 \mod 4.$$

[1] big and nef

[2] Cfr. Garbagnati-Sarti, Sarti-van Geemen and then Huybrechts book

- We have $2M \sim N_1 + \cdots + N_8$ with $N_k = \mathbf{P}^1$ and $N_i N_j = -2\delta_{ij}$.

- $N := N_1 + \cdots + N_8$ defines the double covering $\pi' : \tilde{S}' \to S$ branched on N and the commutative diagram

$$\begin{array}{ccc} \tilde{S}' & \xrightarrow{\nu'} & \tilde{S} \\ {\scriptstyle \pi'}\downarrow & & {\scriptstyle \pi}\downarrow \\ S & \xrightarrow{\nu} & \overline{S} \end{array}$$

ν is the contraction of N [3] and $\tilde{S}$ is a minimal K3 surface.

- π is the quotient map of a symplectic involution $\iota : \tilde{S} \to \tilde{S}$ branched exactly on the even set of nodes

$$\{o_1 := \nu(N_1)\ ,\ \ldots\ ,\ o_8 := \nu(N_8)\} = \mathrm{Sing}\,\overline{S}.$$

[3] Let $E_i = \pi'^{-1}(N_i)$, $i = 1 \ldots 8$, then E_i is an exceptional line on the smooth surface $\tilde{S}'$. It turns out that ν' is the contraction of $E_1 + \cdots + E_8$

- Notations:
  - $\mathcal{F}_g :=$ moduli of genus g K3 surfaces $(S, \mathcal{C})$,
  - $\mathcal{F}_g^N :=$ moduli of genus g Nikulin surfaces $(S, \mathcal{C}, \mathcal{M})$.
- For a general $[S, \mathcal{C}, \mathcal{M}] \in \mathcal{F}_g^N$ one has

$$\operatorname{Pic} S = \mathbb{Z}[\mathcal{C}] \perp \mathbb{L}_S,$$

  where $\mathbb{L}_S$ is generated by $\mathcal{M}, \mathcal{O}_S(N_1), \dots, \mathcal{O}_S(N_8)$. [4]
- With a slight abuse we can say that

$$\mathcal{F}_g^N \subset \mathcal{F}_g.$$

[4] As an abstract lattice $\mathbb{L}_S$ is known as the Nikulin lattice.

- An intermediate divisor in $\mathcal{F}_g$:

$$\mathcal{D}_g := \{\ [S, \mathcal{C}] \in \mathcal{F}_g \ /\ \exists\ \mathcal{M} \in \operatorname{Pic} S,\ < \mathcal{M}, \mathcal{C} >= 0\}$$

  so that $< \mathcal{M}, \mathcal{M} >= -4$. We assume $\mathcal{C} \otimes \mathcal{M}^{-1}$ big and nef.

- For a general $[S, \mathcal{C}] \in \mathcal{D}_g$ the element $\mathcal{M}$ is unique and

$$\operatorname{Pic} S = \mathbb{Z}[\mathcal{C}] \perp \mathbb{Z}[\mathcal{M}].$$

- Clearly:

$$\mathcal{F}_g^N \subset \mathcal{D}_g \subset \mathcal{F}_g.$$

- In low genus $\mathcal{F}_g^N$ sits in a fascinating system of relations to other geometric families. We present some work in progress about. [5]
- For $g \leq 10$ it seems interesting to study Mukai constructions for a Nikulin surface.
- The unirationality of $\mathcal{F}_g^N$ is known for $g \leq 7$ [6] We prove here:

[5] Part of it jointly with A. Garbagnati

[6] Farkas-Verra to appear in Advances of Math.

- Theorem (1)

$$\mathcal{F}_8^N \text{ is rational.}$$ [7]

- Theorem (2)

$$\mathcal{D}_8 \text{ is birational to } \mathbf{P}^{14} \times \mathcal{P}_6.$$

  - $\mathcal{P}_6$ denotes the moduli space of six unordered points of $\mathbf{P}^2$.
  - Its rationality is an unknown, apparently difficult, problem.
  - A natural question: is $\mathcal{F}_g^N$ rational for $g \leq 7$?

[7] — to appear in *K3 surfaces and their moduli* Proceedings Schirmonnikoog 2014

- There is a beautiful geometry behind theorems 1 and 2 we want to discuss during this talk.
- Further notations for $[S, \mathcal{C}, \mathcal{M}] \in \mathcal{D}_g$, $g \geq 3$:
  - $\mathcal{H} := \mathcal{C}(-M)$ and $\mathcal{A} := \mathcal{C}(-2M)$, moreover
  - $C \in |\mathcal{C}|$, $H \in |\mathcal{H}|$, $A \in |\mathcal{A}|$, [8].
- For a general $[S, \mathcal{C}, \mathcal{M}] \in \mathcal{F}_g^N$ the map $f_{\mathcal{C}} \times f_{\mathcal{H}}$ defines an embedding

$$S \subset \mathbf{P}^g \times \mathbf{P}^{g-2}.$$

- For a general $[S, \mathcal{C}, \mathcal{M}] \in \mathcal{D}_g$, $\mathcal{C}$ and $\mathcal{H}$ are very ample as soon as their genus is ≥ 3.

[8] provided these linear systems are not empty.

- For a general $[S, \mathcal{C}, \mathcal{M}] \in \mathcal{F}_g^N$ we have:
  - $f_{\mathcal{H}}(S) = S$ and $f_{\mathcal{H}}(N_i)$ is a line.
  - $f_{\mathcal{C}}(S) = \overline{S}$ and $f_{\mathcal{C}}(N_i)$ is a node.
- The next characterization of $\mathcal{F}_g^N$ in $\mathcal{D}_g$ is useful: [9]

Proposition

Let $[S, \mathcal{C}, \mathcal{M}] \in \mathcal{D}_g$, the following conditions are equivalent:

- $[\mathcal{S}, \mathcal{C}, \mathcal{M}] \in \mathcal{F}_g^N$,
- $\exists\ N_1 \dots N_8$ *disjoint copies of* $\mathbf{P}^1$ $/$ $HN_i = 1$, $AN_i = 2$.

[9] $g \equiv 0 \mod 4$

- Finally we fix the projective models

$$S \subset \mathbf{P}^{g-2}$$

defined by $\mathcal{H}$ and

- $$\overline{S} \subset \mathbf{P}^{g}$$

defined by $\mathcal{C}$.

- We also recall that C is Prym canonically embedded by $\mathcal{H}$. [10]

- We start with the geometry of $\mathcal{F}_g^N$ for $g \leq 7$.

[10] $\eta_C := \mathcal{O}_C(-M)$ is non trivial of 2-torsion in Pic C so that $\mathcal{O}_C(H) \cong \omega_C \otimes \eta_C$.

- As a kind of nice examples we consider the cases $g = 4, 6, 7$.
- $g = 4$. Let $B \subset \mathbf{P}^4$ be a rational normal quartic and $V := Sec\ B$. One has a quasi-étale double covering

$$\pi : \tilde{V} \to V$$

branched on $B = Sing\ V$ [11]. One can show that:

- Proposition

A general model $\overline{S}$ of genus 4 is a quadratic section of V. [12]

- It follows that $\mathcal{F}_4^N \cong |\mathcal{O}_V(2)|/Aut\ B$ which is rational.

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[11] $\tilde{V} := \{ (x, l) \in B \times V \ / \ x \in l \cap B \}$

[12] In particular $\mathrm{Sing}\ \overline{S} = \mathrm{Sing}\ V \cap \overline{S} = B \cap \overline{S}$.

- $g = 6$. Let $Q \subset \mathbf{P}^4$ be a smooth quadric, the tangential quadratic complex of Q is

$$W := \{l \in G(2,5) \ / \ l \text{ is tangent to } Q\}.$$

- W is endowed with the quasi-étale double covering

$$\pi : \tilde{W} \to W$$

branched on $\operatorname{Sing} W =$ the Veronese embedding of $\mathbf{P}^3$ in $G(1,4) \subset \mathbf{P}^9$. One can show that:

- Proposition

A general model $\overline{S}$ of genus 6 is a linear section of W.

- It follows that $\mathcal{F}_6^N$ is unirational.

- $g = 7$. Consider the model $S \subset \mathbf{P}^5$ defined by $\mathcal{H}$: S is the base locus of a net of quadrics.
- Choosing $N_1 \dots N_7$ it turns out that $C \sim C_o := R + N_1 + \cdots + N_7$, with R a rational normal quintic.
- C_o is the union of R and seven bisecant lines to it.
- Starting from a curve C_o, this curve uniquely defines a net of quadrics and hence its base locus S.
- Moreover S turns out to be a general Nikulin surface of genus 7 endowed with an eighth line disjoint from R.

- Proposition

The moduli space $\tilde{\mathcal{F}}_7^N$ of curves C_o is rational and has a map of degree 8

$$f : \tilde{\mathcal{F}}_7^N \to \mathcal{F}_7^N.$$[13]

[13] Actually $\tilde{\mathcal{F}}_7^N$ is the moduli of fourtuples $(S, \mathcal{C}, \mathcal{M}, N_i)$ such that $(S, \mathcal{C}, \mathcal{M})$ is a Nikulin surface of genus 7 and N_i is one of the lines in $S \subset \mathbf{P}^5$. The rationality of $\mathcal{F}_7^N$ is not clear.

2. Nikulin surfaces of genus 8 and rational normal sextics

- Let $g = 8$ and $[S, \mathcal{C}] \in \mathcal{D}_8$ be general, we have an embedding

$$S \subset \mathbf{P}^6$$

with hyperplane sections $H \sim C - M$ of genus 7.

- For $g = 8$ we have $(C - 2M)^2 = -2$ and $(C - 2M)H = 6$.

- **Proposition**

Let $A \in |C - 2M|$ and $[S, \mathcal{C}, \mathcal{M}] \in \mathcal{F}_8^N$ general. Then A is a smooth, integral rational normal sextic spanning $\mathbf{P}^6$. [14]

- **Proposition**

For a general $[S, \mathcal{C}, \mathcal{M}] \in \mathcal{F}_8^N$ the lines $N_1 \dots N_8$ are disjoint bisecant lines to A contained in S.

[14] Then the same is true by semicontinuity on $\mathcal{D}_8$.

- The Mukai-Brill-Noether theory is known for $[X, \mathcal{O}_X(1)] \in \mathcal{F}_6$[15]:
  - CASE 1:
  - If a smooth $H \in |\mathcal{O}_X(1)|$ is not trigonal nor biregular to a plane quintic, then H is generated by quadrics.
  - $\exists!$ H-stable rank 2 vector bundle $\mathcal{E}$ on X such that:

    (i) *det* $\mathcal{E} \cong \mathcal{O}_X(1)$;

    (ii) $h^0(\mathcal{E}) = 5$ *and* $h^i(\mathcal{E}) = 0$ *for* $i \geq 1$;

    (iii) *det* : $\wedge^2 H^0(\mathcal{E}) \to H^0(\mathcal{O}_{\mathbf{P}^6}(1))$ *is surjective.*

[15] For simplicity we assume that $\mathcal{O}_X(1)$ is very ample

- Let $G(1,4) \subset \mathbf{P}^9 := \mathbf{P} \wedge^2 H^0(\mathcal{E})^*$ be the Plücker embedding of the Grassmannian of lines of $\mathbf{P}H^0(\mathcal{E})^*$. Then the diagram

$$\begin{tikzcd} \mathbf{P}^6 \arrow[r, "\delta"] & \mathbf{P}^9 \\ X \arrow[u] \arrow[r, "f_{\mathcal{E}}"] & G(1,4), \arrow[u] \end{tikzcd}$$

commutes, where $\delta := det^*$, the vertical maps are the inclusions and $f_{\mathcal{E}}$ is the embedding defined by $\mathcal{E}$.

- Up to obvious identifications we can say that

$$X \subset T := \mathbf{P}^6 \cap G(1,4) \subset \mathbf{P}^9.$$

- Mukai theory in genus 6 says also that:

  (iv) *X is a quadratic section of T,*

- Since X is a smooth quadratic section of T, T is an integral threefold linear section of $G(1,4)$ with isolated singularities.

- Actually T is a smooth Del Pezzo threefold of degree 5 if X is sufficiently general.

- - CASE 2:
  - Assume H is either trigonal or biregular to a plane quintic. Then H has Clifford index 1 and the following property holds true:
  - there exists an integral curve $D \subset X$ such that either $DH = 3$ and $D^2 = 0$ or $DH = 5$ and $D^2 = 2$.
- A general genus 8 Nikulin surface occurs in case (1), not in (2).

- Proposition

Let $S \subset \mathbf{P}^6$ be a general Nikulin surface of genus 8 embedded by $f_{\mathcal{H}}$. Then S is a quadratic section of a threefold T as above. [16]

[16] PROOF Pic S is the orthogonal sum of rank 9 $\mathbb{Z}\mathcal{L} \oplus \mathbb{L}_S$, where $\mathbb{L}_S$ is the Nikulin lattice generated by $\mathcal{O}_S(M), \mathcal{O}_S(N_1) \ldots \mathcal{O}_S(N_8)$. A standard computation we omit, shows that no divisor D exists such that $D^2 = 0$ and $DH = 3$ or $D^2 = 2$ and $DH = 5$. This excludes case (2).

- With A and $S \subset T = \mathbf{P}^6 \cap G(1,4) \subset \mathbf{P}^9$ as above, qnd under the previous generality assumptions, we study the restriction

$$\mathcal{E}_A := \mathcal{E} \otimes \mathcal{O}_A$$

of the Mukai bundle $\mathcal{E}$ and discuss the possible cases. Of course we have $\mathcal{E}_A = \mathcal{O}_{\mathbf{P}^1}(m) \oplus \mathcal{O}_{\mathbf{P}^1}(n)$ with $m + n = 6$.

Proposition

One has $m, n \geq 0$ so that $h^0(\mathcal{E}_A) = 8$ and $h^1(\mathcal{E}_A) = 0$.

[17]

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[17] PROOF Consider the commutative diagram

$$\begin{array}{ccc} \wedge^2 H^0(\mathcal{E}) & \xrightarrow{\wedge^2 r} & \wedge^2 H^0(\mathcal{E}_A) \\ {\scriptstyle det}\downarrow & & {\scriptstyle det_A}\downarrow \\ H^0(det\ \mathcal{E}) & \xrightarrow{r} & H^0(det\ \mathcal{E}_A) \end{array}$$

The restriction r is an isomorphism and det is surjective. This implies $m, n \geq 0$: otherwise det_A would be zero.

- Now we consider the tautological map

$$u_A : \mathbb{P}_A \to \mathbf{P}^7 := \mathbf{P}H^0(\mathcal{E}_A)^*$$

  of the ruled surface $\mathbb{P}_A := \mathbf{P}\mathcal{E}_A^*$.
- Then

$$R := u_A(\mathbb{P}_A).$$

  is a rational normal scroll of degree 6.
- The next standard exact sequence will be crucial:

$$0 \to \mathcal{E}(-A) \to \mathcal{E} \to \mathcal{E}_A \to 0.$$

- The associated long exact sequence is the following:

$$0 \to H^0(\mathcal{E}) \to H^0(\mathcal{E}_A) \xrightarrow{\delta_A} H^1(\mathcal{E}(-A)) \to 0.$$

- In particular one has
  - $h^0(\mathcal{E}) = 5$,
  - $h^0(\mathcal{E}_A) = 8$,
  - $h^1(\mathcal{E}(-A)) = 3$. [18]

- The coboundary map $\partial_A : H^0(\mathcal{E}_A) \to H^1(\mathcal{E}(-A))$ defines a plane

$$P_A := \mathbf{P} Im\, \partial_A^* \subset \mathbf{P}^7.$$

[18] PROOF Since $\mathcal{E}(-A)$ is H-stable and $H(H-2A) < 0$, it follows $h^0(\mathcal{E}(-A)) = 0$. Furthermore we know that $h^i(\mathcal{E}) = 0$ for $i \geq 1$ and we have $h^1(\mathcal{E}_A) = 0$ because $m, n \geq 0$. This implies the statement.

- Let $\mathbf{P}^4 := \mathbf{P}H^0(\mathcal{E})^*$. Dualizing the sequence and projectivizing we define the linear projection of center P_A:

$$\alpha_A : \mathbf{P}^7 \to \mathbf{P}^4 := \mathbf{P}H^0(\mathcal{E})^*.$$

- Furthermore we have the commutative diagram

$$\begin{array}{ccc} \mathbf{P}^7 & \xrightarrow{\alpha_A} & \mathbf{P}^4 \\ {\scriptstyle u_A}\uparrow & & \uparrow{\scriptstyle u_S} \\ \mathbf{P}\mathcal{E}_A^* & \xrightarrow{i} & \mathbf{P}\mathcal{E}^* \end{array}$$

  where the vertical arrows are the tautological maps.
- The constructions holds true for a general element of $\mathcal{D}_8$, not only in the Nikulin case.
- We will profit of this construction in the next section, where the very special feature of the projection α_A will be described.

- Let $G(1,7)$ be the Grassmannian of lines of $\mathbf{P}H^0(\mathcal{E}_A)^*$. Then

$$l \longrightarrow \alpha_A(l), \quad l \in G(1,7)$$

defines a linear projection

$$\lambda_A : G(1,7) \to G(1,4).$$

- The next diagram is commutative:

$$\begin{array}{ccc} G(1,7) & \xrightarrow{\lambda_A} & G(1,4) \\ {\scriptstyle f_{\mathcal{E}_A}}\Big\uparrow & & {\scriptstyle f_{\mathcal{E}}}\Big\uparrow \\ A & \xrightarrow{i} & S \end{array}$$

Theorem

For a general Nikulin surface of genus 8 one has

$$\mathcal{E}_A = \mathcal{O}_{\mathbf{P}^1}(3) \oplus \mathcal{O}_{\mathbf{P}^1}(3).$$

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[19] PROOF We have $\mathcal{E}_A = \mathcal{O}_{\mathbf{P}^1}(m) \oplus \mathcal{O}_{\mathbf{P}^1}(n)$ with $0 \leq m \leq n \leq 6$ and $m + n = 6$. It suffices to show that R is not a cone and that no rational section of degree 1 or 2 is contained in it. This implies $m = 3$. To this purpose consider the projected scroll $R' = \alpha_A(R)$. Since A is embedded in $G(1,4)$ as an integral sextic curve, the degree of R' is six. For any integral variety $Y \subset \mathbf{P}^4$ we denote by σ_Y the variety in $G(1,4)$ parametrizing the lines intersecting Y. Let us exclude the cases $0 \leq m \leq 2$. $m = 0$. Then the scroll R' is a cone of vertex o and A is contained in σ_o. But σ_o is a linear space of dimension four and A would be a degenerate curve in it, which is excluded. $m = 1$. In this case R' contains a line L intersecting every line of its ruling. Consider σ_L: it is well known that σ_L is a cone of vertex a point l over the Segre embedding $\mathbf{P}^1 \times \mathbf{P}^2 \subset \mathbf{P}^5$. Since $A \subset \sigma_L$ it follows that $\sigma_L \subset \mathbf{P}^6 =< A >$. Moreover $\mathbf{P}^6$ is the linear space tangent to $G(1,4)$ at the parameter point of L. But then $T = \sigma_L$: a contradiction. $m = 2$. We can assume that R' contains a smooth conic K intersecting all the lines of the ruling of R'. Let P be the supporting plane of K, then S is contained in the codimension 1 Schubert cycle σ_P. This is endowed with a ruling of 4-dimensional smooth quadrics having the dual plane P^* as the base locus. Every element of such a ruling is the Plücker embedding of the Grassmannian of the lines contained in a hyperplane through P. Notice also that *Sing* $\sigma_P = P^*$. Then, since S is a smooth complete intersection of three hyperplane sections of $G(1,4)$ and of a quadric section, it follows that $S \cap P^* = \emptyset$. But then this ruling of quadrics of σ_P cuts on S a base pont free pencil $|D|$ such that $D^2 = 0$ and $DH = 4$. This is excluded again by a standard computation in the Picard lattice of a general Nikulin surface.

3. Nikulin surfaces of genus 8 and symmetric cubic threefolds

- A symmetric cubic threefold is a cubic hypersurface

$$V := \{det(a_{ij}) = 0\} \subset \mathbf{P}^4,$$

where $a_{ij} = a_{ji}$ are linear forms.

- We assume dim $< a_{ij} >= 6$ so that $V = Sec\ B$, B a rational normal quartic curve.

- The family of bisecant lines to B is a 3-Veronese embedding

$$W \subset G(1, 4)$$

embedded as a congruence of class $(3, 6)$.

- Since $\mathcal{E}_A$ is balanced then $\mathbf{P}\mathcal{E}_A^* = \mathbf{P}^1 \times \mathbf{P}^1$ and

$$R := u_A(\mathbf{P}\mathcal{E}_A^*) \subset \mathbf{P}^7 = \mathbf{P}H^0(\mathcal{E}_A)^*$$

  is the image of $|\mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(1,3)|$.
- R is a rational normal sextic scroll: we fix it once at all.
- Restricting to R the top arrow of the previous diagram

$$\begin{tikzcd}
\mathbf{P}^7 \arrow[r, "\alpha_A"] & \mathbf{P}^4 \\
\mathbf{P}\mathcal{E}_A^* \arrow[u, "u_A"] \arrow[r, "i"] & \mathbf{P}\mathcal{E}^* \arrow[u, "u_S"]
\end{tikzcd}$$

- We obtain a linear projection

$$\alpha : R \to \mathbf{P}^4.$$

- α is a finite morphism of degree 1 onto its image. Let

  $$Z \subset R$$

  be the subscheme of points where α is not an embedding. Then

  $$\ell(Z) = 12$$

  by double point formula.

- In other words R has six apparent ordinary double points if α is a sufficiently general projection in $\mathbf{P}^4$.

- This is actually not the case for simple geometric reasons:
- A has 8 bisecant lines $N_1 \dots N_8 \subset S \subset G(1,4)$ in the Nikulin case,
- $\alpha(R)$ is the projection in $\mathbf{P}^4$ of the universal line over A:

$$\mathbf{P}_A := \{(x,l) \in \mathbf{P}^4 \times G(1,4) \ / \ x \in l\},$$

- N_i parametrizes a pencil of lines in $\mathbf{P}^4$ of center say n_i,
- the fibre of $\mathbf{P}_A$ at $N_i \cap A$ is the disjoint union of two lines of N_i.
- Hence:

$$\operatorname{Sing} \alpha(R) \supseteq \{n_1 \dots n_8\} \ !$$

- Theorem
  - $\mathrm{Sing}\,\alpha(R)$ *is a rational normal quartic* B,
  - $\alpha(R)$ *is a fake K3 surface of genus 4:*
  - *let* $V = Sec\ B$ *then*

$$\alpha(R) = Q \cap V, \quad Q \in |\mathcal{I}_{B/\mathbf{P}^4}(2)|.$$

- So far A is defined by a special embedding $\alpha : \mathbf{P}^1 \to G(1,4)$ of degree 6:
- $< A > \cap G(1,4) = T$ as for every rational normal sextic,
- but $A = W \cap T$, where $W = \mathbf{P}^2$ embedded with class $(3,6)$.
- A special feature: A has a 1-dimensional family

$$E_A := \{\text{lines } N \text{ such that } N \subset A \cap G(1,4)\}.$$

- More geometry of the special embeddings $\alpha : \mathbf{P}^1 \to A \subset G(1,4)$:
- The family E_A is an elliptic curve.[20]
- A defines a second fake surface of genus 6, namely

$$S_A = \bigcup N, \ \ N \in E_A.$$

- $S_A \in |\mathcal{I}_{A/T}(2)|$, actually $\mathrm{Sing}_A = A$.

[20] Naturally embedded as a curve of type (2,2) in $B \times B$.

- The family of special embeddings α modulo $Aut(G(1,4))$ is

$$|\mathcal{I}_{B/V}(2)|/Aut\ B = |\mathcal{O}_{\mathbf{P}^1[2]}(2)|/Aut\ \mathbf{P}^1$$

that is a rational surface we will denote by

$$\Sigma.$$

- The considered Nikulin surface S belongs to $|\mathcal{I}_{A/T}(2)|$.
- A general $S' \in |\mathcal{I}_{A/T}(2)|$ is a smooth Nikulin surface.
- Proof: $S' = Q' \cap T$ and

$$Q' \cdot S_A = 2A + N'_1 + \cdots + N'_8,$$

N'_i a bisecant line to A.

- Let $\alpha \in \mathbf{P}^5 := |\mathcal{I}_{B/V}(2)|$, we denote by $\alpha : \mathbf{P}^1 \to G(1,4)$ the corresponding sextic embedding and put $A = \alpha(\mathbf{P}^1)$:
- From the previous remarks and construction one has a $\mathbf{P}^9$-bundle

$$\pi : \mathbb{P} \to \mathbf{P}^5 \quad (^{21})$$

with fibre at α the linear system of Nikulin surfaces $|\mathcal{I}_{A/T}(2)|$.

- With some more elaboration:
  - The natural map $\mathbb{P}/Aut\ B \to \mathcal{F}_8^N$ is birational.
  - $\mathbb{P}/Aut\ B$ is birational to $\mathbf{P}^9 \times \Sigma$.

[21] onto a non empty open set of

- We have sketched the proof that

Theorem

The moduli space of genus 8 Nikulin surfaces is rational.

- A Mukai construction, in some sense, for the model $\overline{S}$ of a general Nikulin surface of genus 8 seems also available:

- Consider the rational map

$$f : T \to \mathbf{P}^9$$

defined by $\mathcal{I}_A(2)$. Let $\overline{T}$ be the birational image of f, possibly:

$$\overline{T} = \mathbf{P}^9 \cap G(1,5).$$

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[22] Work in progress

- f contracts S_A to a copy of the elliptic curve E_A spanning a hyperplane:
- $Sing\ \overline{T} = E_A$.
- Which cubic 3-fold is 'naturally' birational to the Fano 3-fold $\overline{T}$? It should be $Sec\ B$.
- One hopes for further progress on Mukai realizations of Nikulin surfaces for $g = 9, 10$[23]

[23] Work in progress

4. Rational normal sextics, 6-nodal cubic 3-folds and $\mathcal{D}_8$

- It is time to consider a general $[S, \mathcal{C}, \mathcal{M}] \in \mathcal{D}_8$:
- the construction considered for $[S, \mathcal{C}, \mathcal{M}] \in \mathcal{F}_8^N$ yelds

$$A \subset S \subset T = \mathbf{P}^6 \cap G(1,4) \subset \mathbf{P}^9,$$

- but this time the commutative diagram

$$\begin{array}{ccc} \mathbf{P}^7 & \xrightarrow{\alpha_A} & \mathbf{P}^4 \\ {\scriptstyle u_A}\uparrow & & \uparrow{\scriptstyle u_S} \\ \mathbf{P}\mathcal{E}_A^* & \xrightarrow{i} & \mathbf{P}\mathcal{E}^* \end{array}$$

- defines a *generic* linear projection

$$\alpha : R \to \mathbf{P}^4$$

of center the plane P_A.[24]

[24] As above $R = u_A(\mathbf{P}\mathcal{E}_A^*)$ and $\alpha := \alpha_A/R$.

- We have:
  - $\mathrm{Sing}\,\alpha(R) = \{o_1 \dots o_6\}$, six independent points [25],
  - A has exactly six bisecant lines contained in $G(1,4)$.
- Modulo $\mathfrak{s}_6$ it is not restrictive to fix the set

$$O := \{o_1 \dots o_6\}.$$

- $\mathcal{R} := \{\text{sextic rational scrolls } \overline{R} \ / \ \mathrm{Sing}\,\overline{R} = O\}$
- We have:

Theorem

$\mathcal{D}_8$ is birational to a $\mathbf{P}^9$-bundle over $\mathcal{R}/\mathfrak{s}_6$. [26]

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[25] In the unique open $PGL(5)$-orbit

[26] More precisely over a no empty open set of $\mathcal{R}/\mathfrak{s}_6$ with fibre $|\mathcal{I}_{A/T}(2)|$ at $\overline{R} = \alpha_A(R)$.

- What is $\mathcal{R}/\mathfrak{s}_6$?

- The answer relies on the studies of a person of the same countryside of Alberto Collino:

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- Corrado Segre (Saluzzo 1863 - Torino 1924) described singular cubic 3-folds in a famous Memoir.

- Here we come to meet six nodal cubic 3-folds V and their geometry described by Corrado Segre, we can assume

$$V \in |\mathcal{I}^2_{\mathcal{O}}(3)|.$$

For this geometry see I. Dolgachev's contribution to the volume

From classical to modern Algebraic Geometry

Corrado Segre's Mastership and Legacy

- To appear in Trends in History of Science, Birkhauser.

- $\mathbb{I} := |\mathcal{I}^2_O(3)|$ is a linear system of dimension 4.
- It defines a *not dominant* rational map $f : \mathbf{P}^4 \to \mathbb{I}^*$.
- $f(\mathbf{P}^4) = \Sigma :=$ Segre's 10-nodal cubic.
- general fibre of f: a rational normal quartic.

- The Fano surface $F(V)$ splits as follows. Let $l \in F(V)$:

$$f(l) = \text{ cubic curve in } f(V) = \text{ smooth cubic surface}$$

- $f(l)$ is either a plane or a skew cubic in $f(V)$.
- Let $f(l)$ be skew: $|f(l)|$ contracts a sixer e of lines.
- $F(V) = f(V) \cup |f(l)| \ \cup \ |5f(l) - 2e|$.
- This configures a *Schlaefli double six*: $|f(l)|$ contracts e and its conjugate sixer is contracted by $|5f(l) - e|$.

- Concerning us and $\mathcal{R}/\mathfrak{s}_6$ it is easy to show that:
    - Let $\overline{R} \in \mathcal{R}$ be general ten $\exists\,!\; V \in \mathbb{I} := |\mathcal{I}^2_{\mathcal{O}}(3)| \;/\; V \supset \overline{R}$.
    - The assignement $\overline{R} \longrightarrow V$ defines a dominant map

$$p : \mathcal{R}/\mathfrak{s}_6 \to \mathbb{I}/\mathfrak{s}_6.$$

- It is well known that $\mathbb{I}/\mathfrak{s}_6$ is rational and actually the weighted projective space of dimension four

$$\mathbb{P}[1,2,3,4,5]$$

- It is the moduli space of Schlaefli double sixers via $F(V)$ and the assignement $V \to f(V)$.

- It admits an obvious double cover

$$\pi : \mathcal{P}_6 \to \mathbb{P}[1,2,3,4,5]$$

where $\mathcal{P}_6$ is the moduli space of six points in $\mathbf{P}^2$.

- The ruling of $\overline{R}$ defines a curve in $F(V)$.

- This is a conic in $|f(l)| \cup |5f(l) - 2e|$, hence

$$\{\overline{R} \in \mathcal{R} / \overline{R} \subset V\} = \mathbf{P}^5 \ V \ \mathbf{P}^5$$

- It follows that $\mathbb{R}/\mathfrak{s}_6$ is birationally a $\mathbf{P}^5$-bundle on $\mathcal{P}_6$: [27]

$$\mathcal{D}_8 \cong (\mathcal{P}_6 \times \mathbf{P}^5) \times \mathbf{P}^9.$$

[27] $l \subset \overline{R}$ is a line of the ruling. The two $\mathbf{P}^5$'s are the spaces of conics of $|f(l)|$ and $|5f - 2e|$.