ON THE RATIONALITY OF THE MODULI OF HIGHER SPIN CURVES IN LOW GENUS

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ABSTRACT: The global geometry of the moduli spaces of higher spin curves and their birational classification is largely unknown for \( g \geq 2 \) and \( r > 2 \). Using quite related geometric constructions, we almost complete the picture of the known results in genus \( g \leq 4 \) showing the rationality of the moduli spaces of even and odd 4-spin curves of genus 3, of odd spin curves of genus 4 and of 3-spin curves of genus 4.

KEY WORDS: Rationality, Higher spin curves, Higher theta-characteristics, Low genus.

MATHEMATICS SUBJECT CLASSIFICATION (2010): 14H10, 14H45, 14E05, 14E08.

1. INTRODUCTION

Let \( C \) be a smooth, irreducible complex projective curve of genus \( g \), a theta characteristic on \( C \) is a square root \( \eta \) of the canonical sheaf \( \omega_C \). By definition a pair \((C, \eta)\) is a spin curve. It is said to be even or odd according to the parity of \( h^0(\eta) \). Starting from Cornalba’s paper [C], the moduli space \( S_g \) of spin curves of genus \( g \) and its compactifications became object of systematic investigations. As is well known \( S_g \) is split in two irreducible connected components \( S_g^+ \) and \( S_g^- \). They respectively correspond to moduli of even and odd spin curves. The Kodaira dimension of \( S_g^\pm \) is completely known, as well as several facts about rationality or unirationality in low genus. The picture is as follows for even or odd spin curves:

\( S_g^+ \) is uniruled for \( g \leq 7 \),
\( S_g^+ \) has Kodaira dimension zero,
\( S_g^+ \) is of general type for \( g \geq 9 \),
\( S_g^- \) is uniruled for \( g \leq 11 \),
\( S_g^- \) is of general type for \( g \geq 12 \).

Moreover the unirationality of \( S_g^- \) and \( S_g^+ \) has been proved respectively for \( g \leq 8 \) and \( g \leq 6 \). Concerning the rationality problem, \( S_g^\pm \) is classically known to be rational for \( g \leq 3 \), while the rationality of \( S_g^\pm \) is a recent result. For more details on the above picture see [F], [FV1], [FV2], [TZ], [V].

Higher spin curves generalize spin curves. By definition a higher spin curve of genus \( g \) and order \( r \) is a pair \((C, \eta)\) such that \( \eta \otimes r \cong \omega_C \). We will also say that \((C, \eta)\) is an \( r \)-spin curve of genus \( g \). The moduli spaces of these pairs are denoted by \( S_g^{1/r} \). They were constructed by Jarvis in [J1] and then studied by several authors, see for instance [CCC], [Ch], [J3].

Concerning the irreducibility of these spaces, it is useful to recall since now how they behave: \( S_g^{1/r} \) is irreducible if \( r \) is odd and \( g \geq 2 \), while \( S_g^{1/r} \) is split in two irreducible connected components if \( r \) is even and \( g \geq 2 \), [I2]. They are distinguished by the condition that \( \eta^{1/r} \) is an
even or odd theta characteristic. However, with the exception of the case of genus 1, the global geometry of \( S_g^{1/r} \) appears to be largely unknown for \( r > 2 \).

From another side a natural, elementary, remark is that for every curve \( C \) the canonical sheaf \( \omega_C \) not only admits square roots, but the roots of order \( 2g - 2 \) and \( g - 1 \) as well. Restricting to \( g - 1 \) roots, they form configurations of line bundles of degree two which are worth of being studied.

For \( r = g - 1 \) the forgetful map \( f : S_g^{1/(g-1)} \to M_g \) has degree \((g - 1)^2g\). Since this grows up very fast, it is seems natural to expect that \( S_g^{1/(g-1)} \) becomes of general type after very few exceptions. About this, assume that \( g - 1 \) is even so that \( \eta^{g(g-1)/2} \) is a theta characteristic. Then every irreducible component of \( S_g^{1/(g-1)} \) dominates \( S_g^+ \) or \( S_g^- \) if \( g \) is odd, via the assignement \((C, \eta) \to (C, \eta^{g(g-1)/2})\).

Therefore, in view of the picture on moduli of spin curves, there exist irreducible components of \( S_g^{1/(g-1)} \) of non negative Kodaira dimension as soon as \( g \geq 8 \). In this frame the first unknown case of low genus to be considered is the genus 4 case. Somehow surprisingly this is still an exception. We prove in this note that

**Theorem 1.1.** The moduli space of 3-spin curves of genus 4 is rational.

Let \((C, \eta)\) be a general 3-spin curve of genus 4. The starting point for proving the theorem is the remark that giving \((C, \eta)\) is equivalent to give the unique effective divisor \( t \in |\eta^{g-2}|\). Furthermore, let \( C \) be canonically embedded in \( P^3 \), then \( 3t \) is the complete intersection of two quadrics and a cubic surface. We show that the GIT-quotient \( Q \) of the family of these complete intersections is rational and that there is a natural birational map between \( Q \) and \( S_4^{1/3} \).

Adding up this result to the known picture we obtain the list of cases of genus \( g \leq 4 \) where the rationality of \( S_g^{1/r} \) is proven. Here is the complementary list of unknown cases for \( g \leq 4 \):

- Moduli of 4-spin curves of genus 3.
- Moduli of odd spin curves of genus 4.
- Moduli of 6-spin curves of genus 4.

In particular it seems that the case of odd spin curves of genus 4 was not considered in the literature. Notice also that \( S_g^{1/(2g-2)} \) splits into the union of two components: the moduli of pairs \((C, \eta)\) such that \( \eta^{g-1} \) is an even theta characteristic and the complementary component. We will denote them respectively by \( S_g^{1/(2g-2)^+}, S_g^{1/(2g-2)^-} \).

We will say that \((C, \eta)\) is an even (odd) \( r \)-spin curve if \( \eta^{g} \) is an even (odd) theta characteristic.

In the final part of this paper we almost complete the picture of the known results in genus \( g \leq 4 \). Building on quite related geometric constructions and methods, we prove the following theorems.

**Theorem 1.2.** The moduli space of odd spin curves of genus 4 is rational.

**Theorem 1.3.** The moduli spaces of 4-spin curves of genus 3 are rational.

We have not found evidence to the uniruledness of \( S_g^{1/r} \) in the only two missing cases in genus \( g \leq 4 \), namely for \( S_4^{1/6^+} \) and \( S_4^{1/6^-} \). The same lack of evidence appears for further very low values of \( g \), say \( g \leq 7 \) and \( r \geq 3 \). Already for these cases, it could be interesting to apply some recent results on the structure of the Picard group of the Deligne-Mumford compactification of \( S_g^{1/r} \) to obtain informations on the Kodaira dimension of these spaces, (cfr. for instance [P] and [RY]).
Let $C$ be a stable curve of genus $g$, we denote by $[C]$ its moduli point in $\overline{M}_g$. In the same way we denote by $[C; L_1, \ldots, L_m]$ the moduli point of $(C; L_1, \ldots, L_m)$, where $L_1 \ldots L_m$ are line bundles on $C$ of fixed degrees.

If $L$ is a line bundle on $X$ then $|L|$ denotes the linear system of the divisors of $X$ defined by the global sections of $L$.

Throughout the paper an elliptic curve $E$ is a smooth, connected curve $E$ of genus 1, marked by one point $o$.

2. 3-spin curves of genus 4

Let $(C, \eta)$ be a spin curve of genus $g$ and order $r$. We will assume that $C$ is canonically embedded in $\mathbb{P}^{g-1}$.

Putting $k = \lceil \frac{g-1}{\deg \eta} \rceil + 1$, we have $h^0(\eta^\otimes k) \geq 1$ by Riemann-Roch. This implies that each effective divisor $t \in |\eta^\otimes k|$ satisfies the condition $rt = C \cdot F$, where $F$ is a hypersurface of degree $k$.

If $\deg \eta$ divides $g$ then $\deg t = g$ and we expect that $t$ is isolated, which is equivalent to $h^1(\eta^\otimes k) = 0$.

Let us focus on the case $g = 4$ and $r = 3$. In this situation $C \subset \mathbb{P}^3$ is a genus 4 curve of degree 6 and $t$ is a divisor in the linear system $|\eta^\otimes 2|$. Then $3t$ is a bicanonical divisor and there exists a quadric surface $S$ such that

$$3t = C \cdot S.$$

**Lemma 2.1.** Let $C$ be a general curve of genus 4, then $h^0(\eta) = 0$ for every 3-spin curve $(C, \eta)$.

**Proof.** We can assume that $C = Q \cap F$, where $Q$ is a fixed, smooth quadric and $F$ a cubic surface. Now assume $h^0(\eta) = 1$ for some cubic root $\eta$ of $\omega_C$. Then there exist points $x, y \in C$ such that $x + y \in |\eta|$ and $3x + 3y = C \cdot H$, where $H \in |O_Q(1)|$. Let $F$ be the family of complete intersections $3x' + 3y' = C' \cdot H'$, where $H' \in |O_Q(1)|$ and $C' \in |O_Q(3)|$ is smooth. It is easy to see that the action of $\text{Aut}Q$ on $F$ has finitely many orbits. On the other hand, since $3x + 3y$ is a complete intersection, it follows $\dim |I_{3x+3y}(C)| = 8$, where $I_{3x+3y}$ is the ideal sheaf of $3x + 3y$. But then, since the moduli space of $C$ is 9-dimensional, $C$ is not general: a contradiction.

From now on our spin curve $(C, \eta)$ will be sufficiently general. In particular we fix the following assumptions:

**Assumption 2.2.**

- $C$ is a complete intersection in $\mathbb{P}^3$ of a smooth quadric $Q$ and a cubic $F$,
- for each $x \in C$ one has $h^0(O_C(3x)) = 1$,
- $h^0(\eta) = 0$ so that $h^0(\eta^\otimes 2) = 1$.

The second condition is just equivalent to say that the two $g_1^1$‘s on $C$ have simple ramification. The third one is satisfied iff the unique effective divisor $t \in |\eta^\otimes 2|$ is not contained in any plane.

It is clear that the locus of moduli of pairs $(C, \eta)$ satisfying these assumptions is a dense open subset of $S_4^{1/3}$. It is also clear from the previous remarks that the bicanonical divisor $3t$ is a complete intersection scheme in the ambient space $\mathbb{P}^3$, namely

$$3t = F \cdot Q \cdot S,$$

where $S$ is a quadric. This defines a second curve, we denote from now on as $E := Q \cdot S$.

We point out that $E$ is uniquely defined by $(C, \eta)$. $E$ is a quartic curve of arithmetic genus one. We will denote by $I_{at}$ the ideal sheaf in $Q$ of the divisor at $C$. Let $o \in t$ be a closed point, we can fix local parameters $x, y$ at $o$ so that $y$ is a local equation of $C$ and $x$ restricts to a local parameter in
Theorem 2.4. \( s \) satisfying 2), 3), 4). We start from a smooth elliptic quartic \( E \) that:

- \( \epsilon \) is a non trivial 3-torsion element of Pic\( ^0 \)E.

Actually the condition that \( E \) be smooth is satisfied as soon as the the pair \((C, \eta)\) is sufficiently general. This is proven in the next theorem, where some useful conditions, satisfied by a general pair \((C, \eta)\), are summarized.

Theorem 2.4. On a dense open set \( U \subset S_{4}^{1/3} \) every point is the moduli point of a spin curve \((C, \eta)\) such that:

1. \((C, \eta)\) is general as in assumption 2.1,
2. \( E \) is a smooth quartic elliptic curve,
3. \( t \) is a smooth divisor of \( E \),
4. \( t \in |\epsilon(1)| \), where \( \epsilon \) is a non trivial third root of \( O_E \).

Proof. We use the irreducibility of \( S_{4}^{1/3} \) when \( r \) is odd and \( g \geq 2 \). \( S_{4}^{1/3} \) is irreducible, so that every non empty open subset of it is dense. Conditions 1) and 2), 3), 4) are open on families of triples \((C, \eta, E)\) hence they define open subsets of \( S_{4}^{1/3} \). We already know that the open set defined by 1) is not empty. Therefore, to prove the theorem, it suffices to produce one pair \((C, \eta)\) satisfying 2), 3), 4). We start from a smooth elliptic quartic \( E \). We have \( E = Q \cdot S \subset P^3 \), where \( Q, S \) are smooth quadrics. Let \( \epsilon \in \text{Pic}^0 \)E be a non trivial element such that \( \epsilon^\otimes 3 \cong O_E \). Since \( \epsilon(1) \) is very ample, a general \( t \in |\epsilon(1)| \) is smooth and not contained in a plane. Note that \( 3t \in |O_E(3)| \). Then, since \( E \) is projectively normal, there exists a cubic surface \( F \) such that

\[
3t = Q \cdot S \cdot F
\]

in the ambient space \( P^3 \). Let \( I_{3t} \) be the ideal sheaf of \( 3t \) in \( Q \), then we have \( h^0(I_{3t}(3)) = 5 \). Moreover the base locus of \( |I_{3t}(3)| \) is \( 3t \). Hence, by Bertini theorem, a general \( C \in |I_{3t}(3)| \) is smooth along \( C - t \). To prove that a general \( C \) is smooth along \( t \) it suffices to produce one element with this property. This is the case for \( E + L \), where \( L \) is a general plane section. Let \( C \in |I_{3t}(3)| \) be smooth and let \( \eta := O_C(1 - t) \). \((C, \eta)\) is a spin curve of order 3 satisfying 2), 3), 4).

\( \square \)

3. Projective bundles related to \( S_{4}^{1/3} \)

Let \((C, \eta)\) be a general spin curve of order 3 and genus 4. We keep the previous conventions, so that \( C \) is canonically embedded in \( P^3 \) as \( Q \cap F \).

It follows from the above theorem that the moduli point \([C, \eta]\) uniquely defines, up to isomorphisms, a triple \((E, \epsilon, t)\) such that \( E \) is a smooth quartic elliptic curve in \( P^3 \) and \( \epsilon \) is a non trivial third root of \( O_E \).

Moreover \( t \) is a smooth element of \(|\epsilon(1)|\) and \( 3t \) is a complete intersection

\[
3t = C \cdot E = F \cdot Q \cdot S \subset P^3,
\]

where \( S \) is a quadric. As a divisor in \( C \), \( t \) is the the unique element of \(|\eta^\otimes 2|\). In order to prove the rationality of \( S_{4}^{1/3} \) our strategy is as follows. We consider the moduli space of elliptic curves \( E \) endowed with a non trivial 3-torsion element of \( \text{Pic}^0 \)E, namely

\[
R_{1,3} := \{ [E, \epsilon] \mid g(E) = 1, \ \epsilon \neq O_E, \ \epsilon^\otimes 3 \cong O_E \}.
\]
Over it we have the moduli space $\mathcal{P}_{1,4}$ of triples $(E, \epsilon, H)$ such that $H \in \text{Pic}^4 E$. This can be also defined via the Cartesian square

$$
\begin{array}{ccc}
\mathcal{P}_{4,1} & \longrightarrow & \mathcal{P}_{4,1}^{\text{ic}} \\
\downarrow & & \downarrow \\
\mathcal{R}_{1,3} & \longrightarrow & \mathcal{M}_1. \\
\end{array}
$$

As usual, $\mathcal{P}_{4,1}^{\text{ic}}$ denotes the universal Picard variety, that is, the moduli space of pairs $(H, E)$ such that $E$ is an elliptic curve and $H \in \text{Pic}^4 E$.

The space $\mathcal{P}_{4,1}$ is a rational surface. Proving its unirationality, so that the rationality follows, is easy. Starting from $\mathcal{P}_{4,1}$ we construct a suitable "tower"

$$
P \xrightarrow{b} P \xrightarrow{a} P_{4,1}
$$

of projective bundles $a, b, c$. Clearly, as a "tower" of projective bundles over a rational base, $P$ is rational. Let $\phi : S^{1/3} \rightarrow \mathcal{P}_{4,1}$ be the rational map defined as follows: $\phi([C, \eta]) := [E, \epsilon]$. Then we will show that $\phi$ factors through a natural birational map between $S^{1/3}$ and $P_c$, so proving that $S^{1/3}$ is rational. In the next subsections we produce the projective bundles which are needed.

3.1. The ambient bundle $P$. Let us start with the universal elliptic curve over $\mathcal{M}_1$ and its pull-back $E \rightarrow \mathcal{R}_{1,3}$. As is well known there exists a Poincaré bundle $P$ on the fibre product $\mathcal{P}_{4,1} \times_U E$, where $U \subset \mathcal{R}_{1,3}$ is a suitable dense open set. In particular the restriction of $P$ to the fibre at $[E, \epsilon, H]$ of the projection map

$$
\alpha : \mathcal{P}_{4,1} \times_U E \rightarrow \mathcal{P}_{4,1}
$$

is given by $P \otimes \mathcal{O}_{([E,\epsilon,H])} \cong H$. Note that $(\alpha_* P)|_{[E, \epsilon, H]} = H^0(H)$ has constant dimension 4.

Let $\mathcal{H} := \alpha_* P$; then, by Grauert’s theorem, $\mathcal{H}$ is a vector bundle of rank 4 over $\mathcal{P}_{4,1}$. We define the the ambient bundle $P$ as follows:

$$
P := \mathcal{P}_{\mathcal{H}}^*.
$$

Its structure map will be denoted as $p : P \rightarrow P_{4,1}$. It is a $\mathbb{P}^3$-bundle over $\mathcal{P}_{4,1}$. In particular the tautological bundle $\mathcal{O}_{\mathbb{P}(1)}$ defines an embedding

$$
\mathcal{P}_{4,1} \times_U E \subset P.
$$

At $x := [E, \epsilon, H]$ this is the embedding $E \subset \mathbb{P}_x = \mathbb{P} H^0(H)^*$ defined by $H$.

3.2. The bundle of quadrics $a : \mathbb{P}_a \rightarrow \mathcal{P}_{4,1}$. Let us consider the map

$$
\mu : \text{Sym}^2 \mathcal{H} \rightarrow \alpha_* (\mathcal{P} \otimes 2)
$$

of vector bundles on $\mathcal{P}_{4,1}$. At $x := [E, \epsilon, H]$ we have $\alpha_* (\mathcal{P} \otimes 2)_x = H^0(H \otimes 2)$ and

$$
\mu_x : \text{Sym}^2 H^0(H) \rightarrow H^0(H \otimes 2)
$$

is the multiplication map. Putting $Q := \ker \mu$ and $\mathbb{P}_a := \mathbb{P} Q$, we denote as

$$
a : \mathbb{P}_a \rightarrow \mathcal{P}_{4,1}
$$

the structure map. The bundle $a$ is a $\mathbb{P}^1$-bundle and the fibre $\mathbb{P}_a$ parametrizes the quadrics containing the tautological embedding $E \subset \mathbb{P}_x$ defined by $H$.  

5
3.3. The $\mathbb{P}^3$-bundle $b : \mathbb{P}_b \to \mathbb{P}_a$. At first we define the $\mathbb{P}^3$-bundle

$$c : \mathbb{P}_c \to \mathbb{P}_{a_1}.$$ 

Its fibre $\mathbb{P}_{c,x}$ will be $[ε \otimes H]$ at $x := [E, ε, H]$. On $\mathcal{P}_{a_1} \times_U \mathcal{E}$ we fix a vector bundle $\mathcal{N}$ whose restriction to the fibre of $\alpha : \mathcal{P}_{a_1} \times_U \mathcal{E} \to \mathcal{P}_{a_1}$ at $x$ is

$$\mathcal{N} \otimes \mathcal{O}_{c,x} \cong ε.$$

The construction of $\mathcal{N}$ is standard: let $β : \mathcal{P}_{a_1} \times_U \mathcal{E} \to \mathcal{R}_{1,3} \times_U \mathcal{E}$ be the natural map. Then we define $\mathcal{N} := β^∗L$, where $L$ is a Poincaré bundle on $\mathcal{R}_{1,3} \times U \mathcal{E}$. Note that $L$ restricted to the fibre at $[E, ε]$ of the projection $γ : \mathcal{R}_{1,3} \times U \mathcal{E} \to \mathcal{R}_{1,3}$ is the line bundle $ε$. We consider the tensor product $\mathcal{H} \otimes \mathcal{N}$ and finally $α_*(\mathcal{H} \otimes \mathcal{N})$. The latter is a rank 4 vector bundle with fibre $H^0(H \otimes ε)$ at $x$. We define

$$\mathbb{P}_b := a^∗Pα_*(\mathcal{H} \otimes ε).$$

$\mathbb{P}_b$ is a $\mathbb{P}^3$-bundle over $\mathbb{P}_a$. The fibre at $x$ of the map $a \circ b : \mathbb{P}_b \to \mathcal{P}_{a_1}$ is the Segre product $|ε \otimes H| \times |\mathcal{I}_E(2)|$, where $\mathcal{I}_E$ is the ideal sheaf of the embedding $E \subset \mathbb{P}_x$.

3.4. The $\mathbb{P}^3$-bundle $c : \mathbb{P}_c \to \mathbb{P}_b$. In the fibre product $\mathbb{P}_b \times_{\mathcal{P}_{a_1}} \mathbb{P}$ we define the following subvarieties

$$t \subset E \subset Q \subset \mathbb{P}_b \times_{\mathcal{P}_{a_1}} \mathbb{P}.$$ 

Let $o \in \mathbb{P}_b \times_U \mathbb{P}$, then $o$ defines a pair $(v, z)$ where $z \in \mathbb{P}_x$ and $x := a \circ b(o) = [E, ε, H]$. Moreover, the point $o$ is an element $t \in |ε \otimes H|$ of the fibre of $\mathbb{P}_b$ at $b(o)$. Finally $b(o)$ is an element $Q \in |\mathcal{I}_E(2)|$, where $\mathcal{I}_E$ is the ideal sheaf of the tautological embedding $E \subset \mathbb{P}_x$. Clearly we have $t \subset E \subset Q$.

The conditions $z \in t, z \in E, z \in Q$ respectively define the closed sets $t, E, Q$. In particular $E$ is a natural embedding of $\mathcal{P}_{a_1} \times_U \mathcal{E}$ in $\mathbb{P}_b \times_{\mathcal{P}_{a_1}} \mathbb{P}$ and $t$ is a Weil divisor in $E$. Let us consider the standard exact sequence

$$0 \to \mathcal{I}_{3t} \to \mathcal{O}_Q \to \mathcal{O}_{3t} \to 0$$

where $\mathcal{I}_{3t}$ is the ideal sheaf of $t$ in $Q$. We pull-back the line bundle $\mathcal{O}_P(3)$ to the fibre product $\mathbb{P}_b \times_{\mathcal{P}_{a_1}} \mathbb{P}$ and tensor the above exact sequence by it. The resulting exact sequence is denoted in the following way:

$$0 \to \mathcal{I}_{3t}(3) \to \mathcal{O}_Q(3) \to \mathcal{O}_{3t}(3) \to 0.$$ 

Let $β : \mathbb{P}_b \otimes \mathbb{P} \to \mathbb{P}_b$ be the projection onto $\mathbb{P}_b$. Then we apply the push-down functor $β_*$ to this new exact sequence. We obtain the exact sequence

$$0 \to β_*\mathcal{I}_{3t}(3) \to β_*\mathcal{O}_Q(3) \to β_*\mathcal{O}_{3t}(3) \to R^1β_*\mathcal{I}_{3t}(3) = 0.$$ 

Here the sheaf $R^1β_*\mathcal{I}_{3t}(3)$ is zero because at any point $p = (t, Q, [E, ε, H]) \in \mathbb{P}_b$ its fibre is $H^1(\mathcal{I}_{3t}(Q)(3)) = 0$. Notice also that the sheaf $F := β_*\mathcal{I}_{3t}(3)$ is a rank 5 vector bundle with fibre $H^0(\mathcal{I}_{3t}(Q)(3))$ at the same point $p$. Finally we define

$$\mathbb{P}_c := PF.$$ 

We denote the structure map of this $\mathbb{P}^4$-bundle as $c : \mathbb{P}_c \to \mathbb{P}_b$. The fibre of $c$ at $p$ is the linear system of cubic sections $C$ of $Q$ containing the scheme $3t \subset E$. Notice that a smooth $C$ is a canonical curve of genus 4 endowed with the order 3 spin structure

$$η := ω_C(−t).$$
4. The rationality of $S^{1/3}_4$

Let $\mathcal{I}_{2t/P^3}$ be the ideal sheaf of $2t \subset C \subset P^3$. Notice also that

**Lemma 4.1.** $|\mathcal{I}_{2t/P^3}(2)|$ is a pencil of quadrics with base locus $E$.

**Proof.** Observe that $\omega_C^\otimes (-2t) \cong \eta^\otimes 2$. Moreover, this is also the sheaf $\mathcal{I}_{2t/C}(2)$. Consider the standard exact sequence of ideal sheaves

$$0 \to \mathcal{I}_{C/P^3}(2) \to \mathcal{I}_{2t/P^3} \to \eta^\otimes 2 \to 0.$$  

Since we have $h^0(\mathcal{I}_{C/P^3}(2)) = h^0(\eta^\otimes 2) = 1$, the statement follows. \hfill \Box

Due to the latter construction there exists a natural moduli map

$$\phi : \mathbb{P}_c \to S^{1/3}_4$$

which sends a point $z = (C, t, Q, [E, \epsilon, H]) \in \mathbb{P}_c$ to the point

$$\phi(z) := (C, \eta),$$

with $\eta = \omega_C(-t)$. Clearly $\phi$ is defined at $z$ if $C$ is smooth. Since $\mathbb{P}_c$ is rational we can finally deduce the rationality of $S^{1/3}_4$, stated in the Introduction. We show that

**Theorem 4.2.** The map $\phi : \mathbb{P}_c \to S^{1/3}_4$ is birational, so that $S^{1/3}_4$ is rational.

**Proof.** At first we show that the map $\phi$ is dominant. Starting with a general point $[C, \eta] \in S^{1/3}_4$ it is possible to reconstruct a point $z = (C, t, Q, [E, \epsilon, H]) \in \mathbb{P}_c$ such that $\phi(z) = [C, \eta]$. Indeed $t$ is the unique element of $|\eta^\otimes 2|$. Then, from the canonical embedding $C \subset P^3$, we reconstruct $E$ as the smooth base locus of the pencil of quadrics $|\mathcal{I}_{2t}(2)|$ considered above. Then we have $H := O_E(1)$ and $\epsilon := H(-t)$. The quadric $Q$ is the unique quadric of $|\mathcal{I}_{2t/P^3}(2)|$ containing $C$. It is clear that $[C, \eta] = \phi(z)$, with $z = (C, t, Q, [E, \epsilon, H])$. Conversely the inverse map of $\phi$ is well-defined too. Starting from a general $[C, \eta]$ the point $z$ is indeed uniquely reconstructed as above. Hence $\phi^{-1}$ is well defined and $\phi$ is birational. \hfill \Box

In the next sections we prove the other rationality results announced in the Introduction.

5. The rationality of $S^{-1}_4$

We start from an odd spin curve $(C, \eta)$ of genus 4. As in the previous sections, $C$ will be sufficiently general. Thus, passing to its canonical model, we have

$$C \subset Q \subset P^4,$$

where $Q = P^1 \times P^1$ is a smooth quadric and $C$ has bidegree $(3, 3)$ in it. Since $\eta$ is odd, there exists a unique $d \in |\eta|$ and we have

$$2d = L \cdot C,$$

where $L$ is a plane section of $Q$ and a conic tritangent to $C$. The condition that both $d$ and $L$ be smooth clearly defines an open set $U \subset S^{-1}_4$. Furthermore it is easily seen that $U \neq \emptyset$. Then, since $S^{-1}_4$ is irreducible, the next lemma follows.

**Lemma 5.1.** For a general $C$ both the divisor $d$ and the conic $L$ are smooth.

Let $o_1, o_2, o_3$ be the three points of $d$. They are not collinear because $h^0(\eta) = 1$. Hence we can fix projective coordinates $(x_0 : x_1) \times (y_0 : y_1)$ on $P^1 \times P^1$ so that

$$o_1 = (1 : 0) \times (1 : 0), \ o_2 = (0 : 1) \times (0 : 1), \ o_3 = (1 : 1) \times (1 : 1).$$

In particular we can assume that these points are in the diagonal

$$L := \{x_0y_1 - x_1y_0 = 0\}$$
of $\mathbb{P}^1 \times \mathbb{P}^1$. Let $\mathcal{I}_{2d}$ be the ideal sheaf of $2d$ in $\mathbb{P}^1 \times \mathbb{P}^1$ and let

$$I := H^0(\mathcal{I}_{2d}(3, 3)).$$

We consider the 9-dimensional linear system $PI$. This is endowed with the map

$$m : PI \to S^1_\omega$$

defined as follows. Let $C \in PI$ be smooth, then $m(C) := [C, \eta]$, where $\eta := \mathcal{O}_C(o_1 + o_2 + o_3)$. It is clear from the construction that $m$ is dominant. Let

(1)

$$G \subset Aut\mathbb{P}^1 \times \mathbb{P}^1$$

be the stabilizer of the set $\{o_1, o_2, o_3\}$. We have:

**Lemma 5.2.** Assume $C_1, C_2 \in PI$ are smooth. Then $m(C_1) = m(C_2)$ if and only if $C_2 = \alpha(C_1)$ for some $\alpha \in G$.

*Proof.* Let $m(C_i) = [C_i, \eta_i]$, $i = 1, 2$. If $m(C_1) = m(C_2)$ there exists a biregular map $a : C_2 \to C_1$. Since $\mathcal{O}_{C_1}(1, 1) \cong \omega_{C_1}$, it follows that $a$ induces an isomorphism $a^* : H^0(\mathcal{O}_{C_1}(1, 1)) \to H^0(\mathcal{O}_{C_2}(1, 1))$. This implies that $a$ is induced by some $\alpha \in Aut\mathbb{P}^1 \times \mathbb{P}^1$. Furthermore, the condition $m(C_1) = m(C_2)$ also implies that $a^*\mathcal{O}_{C_2}(o_1 + o_2 + o_3) \cong \mathcal{O}_{C_1}(o_1 + o_2 + o_3)$. Hence $\alpha \in G$. The converse is obvious. \qed

Now observe that $G$ acts, in the natural way, on $PI$ and that $m : PI \to S^1_\omega$ is dominant. Then, as an immediate consequence of the previous lemma, we have

**Corollary 5.3.** $S^1_\omega$ is birational to the quotient $PI/G$.

Thus the rationality of $S^1_\omega$ follows if $PI/G$ is rational. In order to prove this, we preliminarily describe the group $G$ and its action on $PI$. We recall that the natural inclusion $Aut\mathbb{P}^1 \times Aut\mathbb{P}^1 \subset Aut\mathbb{P}^1 \times \mathbb{P}^1$ induces the exact sequence

$$0 \to Aut\mathbb{P}^1 \times Aut\mathbb{P}^1 \to Aut\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{Z}_2 \to 0,$$

where $\mathbb{Z}_2$ is generated by the class of the projective involution

$$l : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$$

exchanging the factors. From the above exact sequence we have the exact sequence

$$0 \to G_3 \to G \to \mathbb{Z}_2 \to 0.$$

Here $G_3$ denotes the stabilizer of the set $O := \{o_1, o_2, o_3\}$ in $Aut\mathbb{P}^1 \times Aut\mathbb{P}^1$. Since $O$ is a subset of the diagonal $L$, $L$ itself is fixed by $G_3$. In particular it follows that $G_3$ is the diagonal embedding in $Aut\mathbb{P}^1 \times Aut\mathbb{P}^1$ of the stabilizer of $\{o_1, o_2, o_3\}$ in $AutL$. As is very well known, this is a copy of the symmetric group $S_3$.

Now we proceed to an elementary and explicit description of the $G$-invariant subspaces of $PI$. From it the rationality of $PI/G$ will follow. We fix the notation $l := x_0y_1 - x_1y_0$ for the equation of the diagonal $L$. Let

$$R = \oplus_{a,b \in \mathbb{Z}} R_{a,b}$$

be the coordinate ring of $\mathbb{P}^1 \times \mathbb{P}^1$, where $R_{a,b}$ is the vector space of forms of bidegree $a, b$. We can assume that $\iota^* : R \to R$ is the involution such that $\iota^*x_i = y_i, i = 0, 1$. On the other hand let

$$h_1 := x_0(y_1 - y_0) + y_0(x_1 - x_0), \quad h_2 := x_1(y_0 - y_1) + y_1(x_0 - x_1), \quad h_3 := x_0y_1 + x_1y_0,$$

so that $\{l, h_1, h_2, h_3\}$ is a basis of $R_{1,1}$. We can also assume that, for each $\sigma \in G_3$, the map $\sigma^* : R \to R$ is such that $\sigma^*l = l$ and $\sigma^*$ permutes the elements of the set $\{h_1, h_2, h_3\}$. Then we observe that the eigenspaces of $\iota^* : R_{1,1} \to R_{1,1}$ are

$$R^+_{1,1} = \langle l \rangle, \quad R^-_{1,1} = \langle h_1, h_2, h_3 \rangle.$$
This implies that
\[ R_{1,1} = \langle l \rangle \oplus \langle h_1 + h_2 + h_3 \rangle \oplus \langle h_1 - h_3, h_2 - h_3 \rangle \]
where all the summands are \(G\)-invariant. Considering the multiplication map
\[ \mu : \text{Sym}^2 R_{1,1} \to R_{2,2} \]
one can check that
\[ \ker \mu = \langle h_3^2 - l^2 - (h_1 - h_3)(h_2 - h_3) \rangle. \]
Then, putting \( h := h_1 + h_2 + h_3 \) and \( h_{ij} := h_i - h_j \), it is easy to deduce that the eigenspaces of \( \iota^* : R_{2,2} \to R_{2,2} \) decompose as follows:
\[ R_{2,2}^+ = \langle h^2 \rangle \oplus \langle hh_{13}, hh_{23} \rangle \oplus \langle h^2_{13}, h^2_{23} \rangle \oplus \langle h_{13}h_{23} \rangle \]
and
\[ R_{2,2}^- = \langle lh \rangle \oplus \langle lh_{13}, lh_{23} \rangle, \]
where each summand appearing above is \(G\)-invariant. Considering the multiplication by \( l \), we have an injection
\[ \langle l \rangle \oplus R_{2,2} \hookrightarrow I. \]
Its image \( lR_{2,2} \subset I \) is a subspace codimension one. Moreover we have
\[ lR_{2,2}^+ \subset I^- \cup R_{2,2}^- \subset I^+, \]
where \( I^+, I^- \) are the eigenspaces of \( \iota^* : I \to I \). Let us consider
\[ c = x_0x_1(x_0 - x_1) + y_0y_1(y_0 - y_1). \]
Notice that \( c \in I \) and that \( \text{div}(c) \) is \(G\)-invariant. Indeed, \( \text{div}(c) \) is the union of the six lines in the quadric \( Q = P^1 \times P^1 \) passing through the points \( o_1, o_2, o_3 \). Notice also that \( c \) is not in \( lR_{2,2} \), in particular \( I = \langle c \rangle \oplus lR_{2,2} \). Notice also that \( \iota^* c = c \).

Summing all the previous remarks up, we can finally describe the eigenspaces of \( \iota^* : I \to I \) and their decompositions as a direct sum of \(G\)-invariant summands.

**Lemma 5.4.** Let \( I^+, I^- \) be the eigenspaces of \( \iota^* : I \to I \), then we have
\[ \circ \quad I^+ = \langle c \rangle \oplus \langle l^2h \rangle \oplus \langle h_{13}^2, h_{23}^2 \rangle, \]
\[ \circ \quad I^- = \langle lh^2 \rangle \oplus \langle lh_{13}, lh_{23} \rangle \oplus \langle lh_{13}^2, lh_{23}^2 \rangle \oplus \langle h_{13}h_{23} \rangle, \]
where each summand is an irreducible representation of \(G\).

Now it is straightforward to conclude. For instance let us consider
\[ B := PI^+ \times PI^- \]
and then the variety
\[ \mathbb{P} := \{(x, p) \in PI \times B \mid x \in \mathbb{P}_p \} \subset PI \times B, \]
where \( p := (p^+, p^-) \in PI^+ \times PI^- \) and \( \mathbb{P}_p \) denotes the line joining \( p^+ \) and \( p^- \). The variety \( \mathbb{P} \) is endowed with its two natural projections
\[ PI \xleftarrow{\beta} \mathbb{P} \xrightarrow{\alpha} B. \]
Note that \( \beta : \mathbb{P} \to PI \) is birational, since there exists a unique line \( \mathbb{P}_p \) passing through a point in \( PI - (PI^+ \cup PI^-) \). Moreover
\[ \alpha : \mathbb{P} \to B \]
is a \( P^1 \)-bundle structure with fibre \( \mathbb{P}_p \) at the point \( p = (p^+, p^-) \in B \). It is also clear that the action of \(G\) on \( PI \) induces an action of \(G\) on \( \mathbb{P} \) and that
\[ PI/G \cong \mathbb{P}/G. \]
More precisely, the map $\iota^*$ acts as the identity on $B$, since its two factors are projectivized eigenspaces of $\iota^*$. Moreover each fibre $\mathbb{P}_p$ of $\alpha$ is $\iota^*$-invariant. Indeed $\iota^*/\mathbb{P}_p$ is a projective involution with fixed points $p^+, p^-$ on the line $\mathbb{P}_p$.

Note that the induced action of $G_3$ on $B$ is faithful, since the 2-dimensional summands of $I^\pm$ are standard representations of $S_3$. Furthermore $G_3$ acts linearly on the fibres of $\alpha : \mathbb{P} \to B$.

Indeed consider any $\phi \in G_3$ and any $p = (p^+, p^-) \in B$. Then $\phi(\mathbb{P}_p)$ is the line $\mathbb{P}_{\phi(p)}$, where $\phi(p) = (\phi(p^+), \phi(p^-))$. In particular the map $\phi/\mathbb{P}_p \to \mathbb{P}_{\phi(p)}$ is a projective isomorphism. Let

$$\hat{\mathbb{P}} := \mathbb{P}/G_3;$$

the latter remarks imply that $\alpha : \mathbb{P} \to B$ descends to a $\mathbb{P}^1$-bundle

$$\hat{\alpha} : \hat{\mathbb{P}} \to B/G_3,$$

over a non empty open set $U \subset B/G_3$. Now let us consider $\iota \in G$ and the involution $\iota : \mathbb{P} \to \mathbb{P}$ due to the action of $G$ on $\mathbb{P}$. It is clear from the previous construction that $\iota$ descends to an involution

$$\hat{\iota} : \hat{\mathbb{P}} \to \hat{\mathbb{P}}$$

which is fixing each fibre of $\hat{\alpha}$ and acts linearly on it. Passing to the quotient

$$\hat{\mathbb{P}} := \hat{\mathbb{P}}/\langle \iota \rangle,$$

it follows that $\hat{\alpha}$ induces a $\mathbb{P}^1$-bundle structure $\hat{\alpha} : \hat{\mathbb{P}} \to B/G_3$.

**Remark 5.5.** Actually $\hat{\alpha}$ has two natural sections $s^\pm : B/G_3 \to \hat{\mathbb{P}}$. They are defined as follows: let $\bar{p} \in B/G_3$ be the orbit of $p = (p^+, p^-) \in B$. Then the fixed points of $\iota : \mathbb{P}_{\bar{p}} \to \mathbb{P}_{\bar{p}}$ are the orbits $\bar{p}^+, \bar{p}^-$ of $p^+, p^-$. Passing to the quotient by $\iota$ they define two distinguished points $\hat{p}^+, \hat{p}^- \in \hat{\mathbb{P}}_{\bar{p}}$; by definition $\hat{s}^\pm := s^\pm(\bar{p})$.

**Theorem 5.6.** The quotient $\mathbb{P}/G$ is rational.

**Proof.** Since $\mathbb{P}/G \cong \hat{\mathbb{P}}$ and $\hat{\alpha} : \hat{\mathbb{P}} \to B/G_3$ is a $\mathbb{P}^1$-bundle, the preceding remarks imply that $\mathbb{P}/G \cong B/G_3 \times \mathbb{P}^1$. Hence it remains to show the rationality of $B/G_3$. This is now straightforward: we have $B = PI^+ \times PI^-$ and $G_3$ acts linearly on both factors. Considering $B$ as the trivial projective bundle over $PI^+$, it follows that $B/G_3$ is a $\mathbb{P}^5$-bundle over $PI^+/G_3$. The rationality of $PI^+/G_3$ is a standard property. Since $PI^+ = \mathbb{P}^3$, it is easily proven considering the decomposition of $I^+$ as a sum of irreducible representations of $G_3$. Hence $B/G_3$ is rational. $\square$

We have already proved that $S^+_{4^1}$ is birational to $\mathbb{P}/G$. Hence it follows:

**Corollary 5.7.** The moduli space of odd spin curves of genus 4 is rational.

6. **The rationality of $S^+_3$**

The rationality result to be proven in this section naturally relies on the geometry of odd spin curves of genus 4 considered above. To see this relation let us fix from now on a general curve $C$ of genus three and two distinct points $n_1, n_2 \in C$. As is well known, the line bundle $\omega_C(n_1 + n_2)$ defines a morphism $\phi : C \to \mathbb{P}^3$ such that

$$C_n := \phi(C) \subset Q \subset \mathbb{P}^3.$$

Here $Q := \mathbb{P}^1 \times \mathbb{P}^1$ is a smooth quadric and the unique quadric through $C_n$. Moreover $C_n$ is a curve of bidegree $(3, 3)$ in $\mathbb{P}^1 \times \mathbb{P}^1$ with exactly one node $n := \phi(n_1) = \phi(n_2)$, see [GL]. Let $R_1, R_2$ be the lines in $Q$ containing $n$. Then the pull-back by $\phi : C \to Q$ of the divisor $R_i$ is

$$n_1 + n_2 + m_i, \ i = 1, 2$$

where $n_1 + n_2 + m_1 + m_2$ is the canonical divisor of $C$ containing the points $n_1$ and $n_2$. Moreover $|n_1 + n_2 + m_i|$ is the pencil $|\omega_C(-m_j)|$, where $j \neq i$. The condition that $O_C(n_1 + n_2)$ is a theta characteristic is reflected by the projective model $C_n$ as follows:
Lemma 6.1. Let $R_1$ and $R_2$ be the two lines of $Q$ passing through the node $n$. Then the following conditions are equivalent:

(i) $R_1$, $R_2$ are tangent to the branches of $C_n$ at $n$.

(ii) $\mathcal{O}_C(n_1 + n_2)$ is a theta characteristic.

Proof. (i) $\Rightarrow$ (ii): Since $R_1 + R_2 \in |\mathcal{O}_Q(1)|$ it follows $\phi^*(R_1 + R_2) = 3n_1 + 3n_2 \in |\omega_C(n_1 + n_2)|$. Hence $2n_1 + 2n_2$ is a canonical divisor and $\mathcal{O}_C(n_1 + n_2)$ is a theta characteristic. (ii) $\Rightarrow$ (i): Since $2n_1 + 2n_2$ is a canonical divisor then $|2n_i + n_j|$ is a pencil, $j \neq i$. Let $\phi_i : C \to \mathbb{P}^1$ be the map defined by $|2C_i + n_j|$, then $\phi : C \to Q$ is the product map $\phi_1 \times \phi_2$. Therefore, up to reindexing, we have $2n_i + n_j = \phi^* R_i$. Hence $R_i$ is tangent to a branch of $n$. □

Assume now that $[C, \eta]$ is a general point of $\mathcal{S}_3^{1/4}$ so that $\eta^{\otimes 2}$ is an odd theta characteristic on $C$. This is equivalent to say that there exist two distinct points $n_1, n_2 \in C$ such that

$$\mathcal{O}_C(n_1 + n_2) \cong \eta^{\otimes 2} \text{ and } \eta^{\otimes 6} \cong \omega_C(n_1 + n_2) \cong \mathcal{O}_C(3n_1 + 3n_2).$$

Considering the morphism $\phi$ defined by $\eta^{\otimes 6}$, we have as above that its image $C_n \subset Q \subset \mathbb{P}^3$

is a curve with exactly one node $n = \phi(n_1) = \phi(n_2)$ and no other singular point. Now we observe that the linear system $|\eta^{\otimes 6}|$ contains the two distinct elements:

- $3n_1 + 3n_2$, where $n_1 + n_2 \in |\eta^{\otimes 2}|$,
- $2o_1 + 2o_2 + 2o_3$, where $o_1 + o_2 + o_3 \in |\eta^{\otimes 3}|$.

Lemma 6.2. One has $h^0(\eta) = 0$, so that $h^0(\mathcal{O}_C(o_1 + o_2 + o_3)) = 1$.

Proof. If $h^0(\eta) \geq 1$ then $\eta \cong \mathcal{O}_C(p)$ for some point $p \in C$. But then $4p \in |\omega_C|$, which is impossible on a general $C$ of genus 3. Now observe that $\omega_C(-o_1 - o_2 - o_3) \cong \eta$. Since $h^0(\eta) = 0$ it follows $h^0(\mathcal{O}_C(o_1 + o_2 + o_3)) = 1$ by Riemann-Roch. □

Lemma 6.3. The points $o_1, o_2, o_3$ are distinct and $\{o_1, o_2, o_3\} \cap \{n_1, n_2\} = \emptyset$. Moreover one has $2o_1 + 2o_2 + 2o_3 = L \cdot C_n$ where $L \in |\mathcal{O}_Q(1)|$ is smooth.

Proof. It suffices to produce one pair $(C, \eta)$ satisfying the statement. Fix in $\mathbb{P}^2$ five general points $o_1, o_2, o_3, n_1, n_2$ and let $L$ be the conic through them. Consider the linear system $\Sigma$ of all quartics $C$ which are tangent to $L$ at $o_1, o_2, o_3$ and tangent to the line $< n_1, n_2 >$ at $n_1, n_2$. It is easy to check that the general $C \in \Sigma$ is smooth. Let $\eta = \mathcal{O}_C(o_1 + o_2 + o_3 - n_1 - n_2)$, then $(C, \eta)$ satisfies the statement. □

Remark 6.4. As above let $\mathcal{O}_C(n_1 + n_2)$ be an odd theta characteristic and let $C_n \subset \mathbb{P}^1$ be the image of the map defined by $|\omega_C(n_1 + n_2)|$. It follows from the previous discussion that there exists a bijection between the set of square roots $\eta$ of $\mathcal{O}_C(n_1 + n_2)$ and the set of tritangent planes $P$ to $C_n - \{n\}$. This bijection associates to $P$ the line bundle $\eta = \mathcal{O}_C(o_1 + o_2 + o_3 - n_1 - n_2)$, where $P \cdot C_n = 2o_1 + 2o_2 + 2o_3$.

To prove the rationality result of this section we proceed as in the previous one. We fix coordinates $(x_0 : x_1) \times (y_0 : y_1)$ on $Q$ so that $o_1 = (1 : 0 : 1), o_2 = (0 : 1 : 0)$ and $o_3 = (1 : 1 : 1)$. Then we observe that the diagonal $L = \{x_0y_1 - x_1y_0\}$ is tritangent to the previous curve $C_n$ at $o_1, o_2, o_3$ and that $n \in Q - L$. Keeping the notations of the previous section we consider the linear system $PI$. $C_n$ is in the family of the singular elements of $PI$. Let $U := Q - L$ for each $n \in U$ we consider the 4-dimensional linear system $D_n \subset |\mathcal{O}_Q(3, 3)|$ of all curves $D$ of bidegree $(3, 3)$ such that:
(i) $2\alpha_1 + 2\alpha_2 + 2\alpha_3 \subset L \cdot D$,
(ii) $D$ has multiplicity $\geq 2$ at $n$,
(iii) $R_i \cdot D = 3n$ for $i = 1, 2$, where $R_1$ and $R_2$ are the lines of $Q$ through $n$.

(i) implies the inclusion $\mathbb{D}_n \subset PI$. We consider the incidence correspondence
$$
\mathbb{D} := \{(D, n) \in PI \times U \mid D \in \mathbb{D}_n\}
$$
together with its two projection maps
$$
PI \leftarrow^{\pi_1} \mathbb{D} \rightarrow^{\pi_2} U
$$
$\mathbb{D}$ is a $\mathbb{P}^1$-bundle via the map $\pi_2 : \mathbb{D} \to U$. On the other hand the closure of $\pi_1(\mathbb{D})$ is the locus of singular elements of $PI$. Now we define a rational map
$$
m : \mathbb{D} \to S^{1/4}_3
$$
as follows. Let $C_n \in \mathbb{D}_n$ be nodal with exactly one node $n$, so that its normalization $\nu : C \to C_n$ is of genus 3. Let $\eta := \nu^*\mathcal{O}_C(\alpha_1 + \alpha_2 + \alpha_3 - \nu^*n)$, by definition
$$
m(C_n) := [C, \eta].
$$
Note that the group $G$, defined as in the previous section, acts on $\mathbb{D}$ in the natural way. The action of $\alpha \in G$ on $\mathbb{D}$ is the isomorphism $f_\alpha : \mathbb{D} \to \mathbb{D}$ sending $(D, n) \in \mathbb{D}$ to $(\alpha(D), \alpha(n))$. The proof of the next lemma is completely analogous to the proof of Lemma 5.2 and hence we omit it. The corollary is immediate.

**Lemma 6.5.** Let $D_1, D_2 \in \mathbb{D}$. Then $m(D_1) = m(D_2)$ iff there exists $\alpha \in G$ such that $\alpha(D_1) = \alpha(D_2)$.

**Corollary 6.6.** The quotient $\mathbb{D}/G$ is birational to $S^{1/4}_3$.

Finally we can deduce that

**Theorem 6.7.** $S^{1/4}_3$ is rational.

**Proof.** It is easy to see, and it follows from the analysis of the previous section on the action of $G$ on $PI$, that the action of $G$ on $\mathbb{D}$ is faithful and linear between the fibres of $\mathbb{D}$. Hence the $\mathbb{P}^1$-bundle $\pi_2 : \mathbb{D} \to U$ descends to a $\mathbb{P}^1$-bundle $\mathbb{B} \to U/G$, which is just $\mathbb{D}/G$. But $U/G$ is rational, since it is a unirational surface, therefore $\mathbb{B} = \mathbb{D}/G$ is rational. Then, by the previous corollary, $S^{1/4}_3$ is rational. \qed

7. THE RATIONALITY OF $S^{1/4}_3$

Let us recall that, for any smooth curve $C$ and any divisor $e$ of degree two on it, the line bundle $\omega_C(e)$ is very ample iff $h^0(\mathcal{O}_C(e)) = 0$. Let $C$ be a general curve of genus 3 and let $\eta$ be any 4-th root of $\omega_C$. Then $\eta^{\otimes 2}$ is an even theta characteristic. We have considered the case where $\eta^{\otimes 2}$ is odd in the previous section.

From now on we assume that $[C, \eta]$ is in $S^{1/4}_3$, so that $h^0(\eta^{\otimes 2}) = 0$. Then the line bundle $\omega_C \otimes \eta^{\otimes 2}$ is very ample and moreover it defines an embedding of $C$ in $\mathbb{P}^3$ as a projectively normal curve whose ideal is generated by cubics, see [Dol], §6.3. Obviously no quadric contains $C$ and we cannot argue as in the previous section. Though the beautiful geometry of cubic surfaces through $C$ can be used, it is simpler to consider the canonical model of $C$. Hence we assume that $C$ is embedded in $\mathbb{P}^3$ as a general plane quartic.

**Lemma 7.1.**

(i) One has $h^0(\eta^{\otimes 3}) = 1$. Moreover, the unique divisor of $|\eta^{\otimes 3}|$ is supported on three distinct points $o_1, o_2, o_3$.
(ii) There exists exactly one cubic $E$ such that $C \cdot E = 4(o_1 + o_2 + o_3)$. Moreover $E$ is smooth with general moduli.
Proof. We have $h^0(\eta^{\otimes 3}) \geq 2$ iff $h^0(\omega_C \otimes \eta^{\otimes 3}) = 1$. This implies that $\omega_C \otimes \eta^{\otimes 3} \cong \mathcal{O}_C(p)$, for some point $p \in C$ such that $4p \in [\omega_C]$. But then $C$ is not a general curve. To complete the proof of (1) and to prove (2) it suffices to construct a pair $(C, \eta)$ with the required properties. Starting from a smooth cubic $E$ consider three distinct non collinear points $o_1, o_2, o_3$ such that $b := 4(o_1 + o_2 + o_3) \in |\mathcal{O}_E(4)|$. It is standard to check that the linear system of plane quartics with base locus $b$ contains a smooth element $C$: see the analogous argument in the proof of theorem 2.3. Let $\eta := \omega_C(-o_1 - o_2 - o_3)$, then $(C, \eta)$ is the required pair. \hfill $\square$

Furthermore let $H := \mathcal{O}_E(1)$ and, as above, $4(o_1 + o_2 + o_3) = E \cdot C$. Let 
\[ \epsilon := H(-o_1 - o_2 - o_3). \]
Clearly $\epsilon$ is a $4$-th root of $\mathcal{O}_E$. Moreover:

**Lemma 7.2.** The line bundle $\epsilon^{\otimes 2}$ is not trivial.

**Proof.** Assume $\epsilon^{\otimes 2}$ is trivial. Then it follows $2o_1 + 2o_2 + 2o_3 = B \cdot E$, where $B$ is a conic. This implies that $h^0(\eta^{\otimes 2}) = h^0(\mathcal{O}_C(B - 2o_1 - 2o_2 - 2o_3)) = 1$. Hence $\eta^{\otimes 2}$ is an odd theta: a contradiction. \hfill $\square$

Let $d := o_1 + o_2 + o_3 \in |H \otimes \epsilon^{-1}|$ be general, it follows from lemma 7.1 and its proof that the linear system 
\[ |\mathcal{I}_{d}(4)| \]
defines a 3-dimensional family of smooth genus 3 spin curves $(D, \eta_D)$ of order 4, such that $\eta_D^{\otimes 2}$ is an even theta characteristic. Such a family is the family of pairs $(D, \eta_D)$ such that $D$ is a smooth element of $|\mathcal{I}_{d}(4)|$ and $\eta_D = \omega_D(-d)$.

Note that the curves $D$ are general in moduli. Since $S_{3}^{1/4+}$ is irreducible and dominates $\mathcal{M}_{3}$, it follows that a dense open set of it is filled up by points $[D, \eta_D]$ realized as above. We can now use these remarks to prove that $S_{3}^{1/4+}$ is birational to a suitable tower of projective bundles over a rational modular curve.

To this purpose we consider the moduli space $\mathcal{T}$ of triples $(E, H, \tau)$ such that $E$ is an elliptic curve, that is a genus 1 curve $1$-pointed by $o$, $H = \mathcal{O}_E(3o)$ and $\tau \in \text{Pic}^0 E$ is a 4-torsion point whose square is not trivial. We then have:

**Proposition 7.3.** $\mathcal{T}$ is a rational curve.

**Proof.** Observe that, on a smooth plane cubic $E$, a 4-th root of $\mathcal{O}_E$ is a line bundle $\tau := \mathcal{O}_E(t - o)$ such that $o, t \in E$ and moreover \( i) 3o \in |\mathcal{O}_E(1)|, \) \( ii) 4t + 2o \in |\mathcal{O}_E(2)|. \) Indeed these conditions are just equivalent to say that $4t \sim 4o$. Notice also that they are fulfilled iff there exists a conic $B$ such that $B \cdot E = 4t + 2o$. Furthermore, it is easy to see that either $\tau^{\otimes 2}$ is not trivial and $B$ is smooth or $B$ is a double line and $B \cdot E = 2(2t + o)$. Assuming the former case we consider the plane cubic $A + B$, where $A$ is the flex tangent to $E$ at $o$. Let $P$ be the pencil of cubics generated by $E$ and $A + B$, then its base locus is the $0$-dimensional scheme $Z := 4t + 5o \subset E$. Note that $Z$, hence $P$, is unique up to projective equivalence. Let $F \in P$ be smooth, then $F$ is endowed with the line bundles $\tau_F := \mathcal{O}_F(t - o)$ and $H_F := \mathcal{O}_F(1)$. Consider the rational map $m : P \to \mathcal{T}$ defined as follows: $m(F) = [F, H_F, \tau_F]$. The construction implies that $m$ is surjective. Hence $\mathcal{T}$ is rational. \hfill $\square$

Now consider the moduli space $A_1(3)$ of abelian curves endowed with a degree 3 polarization. This is just the moduli space of pairs $(E, H)$. Therefore the curve $\mathcal{T}$ is a finite cover of $A_1(3)$ via the forgetful map 
\[ f : \mathcal{T} \to A_1(3), \]
sending \([E, H, \tau]\) to \([E, H]\). Over suitable open sets we fix the universal family of abelian curves \(E \to A_1(3)\) and a Poincaré sheaf \(\mathcal{P}\) on \(A_1(3) \times M_1, E\). Then the restriction of \(\mathcal{P}\) to the curve \([E, H] \times E\) is the line bundle \(H\). We consider the map

\[ f \times id_E : \mathcal{T} \times M_1, E \to A_1(3) \times M_1, E \]

and the pull-back

\[ \tilde{\mathcal{P}} := (f \times id_E)^* \mathcal{P} \]

of \(\mathcal{P}\) over the surface

\[ \tilde{E} := \mathcal{T} \times M_1, E. \]

The projection \(u : \tilde{E} \to \mathcal{T}\) is an elliptic fibration: its fibre at \([E, H, \tau]\) is the elliptic curve \(E = [E, H, \tau] \times E\). Since \(E\) is 1-pointed by \(o\), the map \(u\) has two sections

\[ s_0, s_1 : \mathcal{T} \to \tilde{E} \]

which are defined as follows. \(s_0\) is the zero section sending \([E, H, \tau]\) to \(o\). On the other hand we define \(t := s_1([E, H, \tau])\) by the condition \(O_E(t - o) \cong \tau\). Let

\[ D_0 := s_0(T), \ D_1 := s_1(T). \]

Over a dense open set of \(\mathcal{T}\) we can finally define the \(\mathbb{P}^2\)-bundles:

\[ \mathbb{T} := \mathbb{P}(u_\ast \tilde{E} \otimes O_E(D_1 - D_0)), \]

\[ \mathbb{P} := \mathbb{P}(u_\ast \tilde{E}^\ast). \]

The fibre of \(\mathbb{T}\) at the point \([E, H, \tau]\) is the linear system \(|H \otimes \tau|\), while the fibre of \(\mathbb{P}\) at the same point is \(\mathbb{P} H^0(H)^\ast\). Now we consider the tautological embedding

\[ \tilde{E} \subset \mathbb{P}. \]

We note that the embedding \(\tilde{E}_e \subset \mathbb{P}_e\) at \(e := [E, H, t] \in \mathcal{T}\), is the embedding \(E \subset \mathbb{P} H^0(H)^\ast\) defined by \(H\). Then we consider the incidence correspondence

\[ Z \subset F := \mathbb{T} \times \tau \mathbb{P} \]

parametrizing the points \([E, H, \tau; d, x] \in \mathbb{T} \times \tau \mathbb{P}\) such that

\[ x \in d \subset E \subset \mathbb{P} H^0(H)^\ast, \]

\[ d \in |H \otimes \tau|, \]

Let \(\pi_1 : F \to \mathbb{T}\) and \(\pi_2 : F \to \mathbb{P}\) be the projection maps, it is clear that

\[ Z \subset \pi_1^\ast \tilde{E} \subset F. \]

Actually \(Z\) is a divisor in \(\pi_1^\ast \tilde{E}\) and the latter, up to shrinking its base, is a smooth family of elliptic curves. Then \(4Z\) is a Cartier divisor in \(\pi_1^\ast \tilde{E}\) and a subscheme of \(F\). Let \(\mathcal{J}\) be its ideal sheaf, from it we obtain a projective bundle

\[ Q := \mathbb{P}\pi_1^\ast(\mathcal{J} \otimes \pi_2^\ast O_E(4)), \]

over a dense open set of \(\mathbb{T}\). Indeed let \(p := [E, H, \tau, d]\) be a general point of \(\mathbb{T}\) and let \(I_{4d}\) be the ideal sheaf of \(4d\) in \(\mathbb{P} H^0(H)^\ast\). Then \(H^0(I_{4d}(4))\) has constant dimension \(4\) and \(Q\) is a \(\mathbb{P}^3\)-bundle over \(\mathbb{T}\) by Grauert’s theorem. Moreover \(Q\) is a \(\mathbb{P}^3\)-bundle over \(\mathbb{T}\), which is a \(\mathbb{P}^2\)-bundle over the rational curve \(\mathcal{T}\). Hence \(Q\) is rational. The conclusion is near: we are going to construct a birational map

\[ m : Q \to S_3^{1/4}. \]

Let us define \(m\): a general point of \(Q\) is a general pair \((p, D)\), where \(p \in \mathbb{T}\) is a point as above and \(D \in Q_p = [I_{4d}(4)]\). By definition \(m(p)\) is the point \([D, \omega_D(-d)]\) of \(S_3^{1/4}\). We conclude that:

**Theorem 7.4.** \(S_3^{1/4}\) is rational.
Proof. Both $\mathbb{Q}$ and $S_4^{1/4+}$ are irreducible of the same dimension. Hence it is enough to show that $m$ is invertible. Let $[C, \eta] \in S_4^{1/4+}$, where $C \subset \mathbb{P}^2$ is a general smooth quartic and $\eta \otimes \mathcal{O}_C(d)$, as above. We know that there exists a unique cubic $E$ such that $E : C = 4d$ and $E$ is general. This defines the point $p = [E, H, \tau, d] \in \mathbb{T}$, where $H := \mathcal{O}_E(1)$ and $\tau := H(−d)$. Moreover $C$ belongs to $\mathbb{Q}_p = [L_{4d}(4)]$. Assume $m$ is not invertible at $[C, \eta]$. Then there exists $D \in \mathbb{Q}_p$ such that $[D, \omega_D(−d)] = [C, \eta]$ and $D \neq C$. But then there exists a linear isomorphism $\alpha : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ such that $\alpha(D) = C$ and $\alpha^*\mathcal{O}_C(d) = \mathcal{O}_D(d)$. Since $h^0(\eta \otimes \mathcal{O}_C) = 1$, it follows $\alpha^*(4d) = 4d$ and $\alpha(E) = E$. Since $E$ is general, $\alpha$ induces a translation or ±1 multiplication on $\text{Pic}^0 E$. But we have $\alpha^* \tau = \tau$ and moreover $\tau \otimes 2$ is not trivial. This implies that $\alpha$ is the identity and $D = C$: a contradiction. □

REFERENCES


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